On the signature of fibre bundles and absolute Whitehead torsion.

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Abstract

In 1957 Chern, Hirzebruch and Serre proved that the signature of the total space of a fibration of manifolds is equal to the product of the signatures of the base space and the fibre space if the action of the fundamental group of the base space on the real cohomology of the fibre is trivial. In the late 1960s Kodaira, Atiyah and Hirzebruch independently discovered examples of fibrations of manifolds with non-multiplicative signature. These examples are in the lowest possible dimension where the base and fibre spaces are both surfaces. W. Meyer investigated this phenomenon further and in 1973 proved that every multiple of four occurs as the signature of the total space of a fibration of manifolds with base and fibre both surfaces. Then in 1998 H. Endo showed that the simplest example of such a fibration with non-multiplicative signature occurs when the genus of the base space is 111.

We will prove two results about the signature of fibrations of Poincaré spaces. Firstly we show that the signature is always multiplicative modulo four, extending joint work with I. Hambleton and A. Ranicki on the modulo four multiplicativity of the signature in a PL-manifold fibre bundle. Secondly we show that if the action of the fundamental group of the base space on the middle-dimensional homology of the fibre with coefficients in $\mathbf{Z}_2$ is trivial, and that the dimension of the base space is a multiple of four, then the signature is multiplicative modulo eight.

The main ingredient of the first result is the development of absolute Whitehead torsion; this is a refinement of the usual Whitehead torsion which takes values in the absolute group $K_1(R)$ of a ring R, rather than the reduced group $\widetilde{K}_1(R)$. When applied to the algebraic Poincaré complexes of Ranicki the "sign" term (the part which vanishes in $\widetilde{K}_1(R)$) will be identified with the signature modulo four. We prove a formula for the absolute Whitehead torsion of the total space of a fibration and a simple calculation yields the first result.

The second result is proved by means of an equivariant Pontrjagin square, a refinement of the usual one. We make use of the Theorem of Morita which states that the signature modulo eight is equal to the Arf invariant of the Pontrjagin square. The Pontrjagin square of the total space of the bundles concerned is expressed in terms of the equivariant Pontrjagin square on the base space and this allows us to compute the Arf invariant.

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Chapter 1

Introduction.

1.1 A brief history of the signature of fibrations.

Given a $4k$-dimensional Poincaré space X the signature is defined as usual to be the signature of the form given by the cup product and evaluation on the fundamental class:

$$H^{2k}(X;\mathbf{C}) \otimes H^{2k}(X;\mathbf{C}) \to \mathbf{C}$$

$$x \otimes y \mapsto \langle [X], x \cup y \rangle$$

(Poincaré spaces will always be oriented and assumed to have the homotopy type of a finite CW-complex in this thesis). If the dimension of X is not a multiple of four the signature is defined to be 0. We denote the signature by $\mathrm{sign}(X)$.

One can easily show that the signature is multiplicative for products of spaces

$$\mathrm{sign}(X \times Y) = \mathrm{sign}(X)\mathrm{sign}(Y)$$

so a natural question to ask is whether this result extends to fibrations of Poincaré spaces.

Definition 1.1. *A (Hurewicz) fibration is a map $p : E \to B$ of topological spaces satisfying the homotopy lifting property; that is for all spaces X and maps $f : X \to E$, $F : X \times I \to B$ such that $F(x,0) = f(x)$ we may find a map $H : X \times I \to E$ such that $H(x,0) = f(x)$ and $p(H(x,t)) = F(x,t)$.*

A slightly better way of seeing the homotopy lifting property is in commutative diagrams. It say that for all commutative diagrams of the type:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x\mapsto(x{,}0)"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"] & B \end{tikzcd}$$

then there exists a map $H : X \times I \to E$ such that:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x \mapsto (x,0)"'] & E \arrow[d, "p"] \\ X \times I \arrow[ur, "H"] \arrow[r, "F"] & B \end{tikzcd}$$

commutes. The space B will always be path-connected in the fibrations we consider. We write $F = p^{-1}(b_0)$, where b_0 is a distinguished base-point of B, for the *fibre* of p. In fact the homotopy type of the space F does not depend on the choice of b_0. We often write $F \to E \to B$ for a fibration.

The most trivial example of a fibration is a product of spaces $X \times Y \to X$. It is known [Got79] that if B and F are n- and m-dimensional Poincaré spaces respectively and that the action of $\pi_1(B)$ on the $H_m(F; \mathbf{Z})$ is trivial (see below) then E is an $(n+m)$-dimensional Poincaré space [Ped82]. Moreover the orientation of E can be made compatible with that of B and F (more about this in chapter 4).

Therefore we may ask: If $F^m \to E^{m+n} \to B^n$ is a fibration of Poincaré spaces (the superscript denoting the dimension of the space), is it true that $\mathrm{sign}(E) = \mathrm{sign}(B)\mathrm{sign}(F)$? In 1957 Chern, Hirzebruch and Serre proved (over afternoon tea at the Institute for Advanced Study in Princeton) that this is the case if the action of the fundamental group of the base-space on the homology of the fibre $H_*(F; \mathbf{C})$ is trivial [CHS57]. We explain what this means:

Let $\omega : I \to B$ be a loop in B starting and finishing at the base-point b_0. Then we have a diagram:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ F \times I \arrow[r, "\omega \circ pr_I"] & B \end{tikzcd}$$

with pr_I and i_0 the obvious projection and inclusion maps. By the homotopy lifting property there exists a map $H : F \times I \to E$ such that

$$\begin{tikzcd} F \arrow[r] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ F \times I \arrow[ur, "H"] \arrow[r, "\omega \circ pr_I"] & B \end{tikzcd}$$

commutes. The composition $F \xrightarrow{f \mapsto (f,1)} F \times I \xrightarrow{H} E$ lands in $F \subset E$ so it is a self-map of F. One can show that it is a homotopy equivalence and that it only depends (up to homotopy) on the class of ω in $\pi_1(B)$. Taking homology this describes the action of $\pi_1(B)$ on $H_*(F)$. The proof of Chern, Hirzebruch and Serre uses the Serre spectral sequence which we do not discuss in depth here, although a certain amount of familiarity with it is assumed. One can use a similar argument to show that the signature is always multiplicative if the dimension of the base space is odd.

The next natural question which could be asked is whether there are any fibrations with non-multiplicative signature. In the late 1960s Kodaira [Kod67], Atiyah [Ati69] and Hirzebruch [Hir69] independently produced examples of surface-bundles over surfaces where the signature of the total space was a multiple of 16. This is the lowest possible dimension where non-multiplicativity could occur. Atiyah went further and derived a characteristic class formula in the case of a differentiable fibre bundle.

In 1972 Meyer [Mey72, Mey73] went on to describe the signature of a fibration over a differentiable manifold as the signature of a local coefficient system: Let M be a $2k$-dimensional manifold and let V be a real vector space equipped with a $(-)^k$-symmetric form. Suppose we have a map $\pi_1(M) \to \mathrm{Aut}(V)$ preserving the form. Then this determines a local coefficient system $\Gamma \to M$ with fibre V. The combination of the cup product and the $(-)^k$-symmetric form on V yield a symmetric form on $H^n(M;\Gamma)$; we refer to the signature of this form as the signature of the local coefficient system. Given a differentiable fibre bundle $F^{2m} \to E^{2n+2m} \to B^{2n}$ with $n+m$ even the action of $\pi_1(B)$ on $H^m(F;\mathbf{R})$ determines a local coefficient system over the base space B. Meyer proved that the signature of this system is equal to the signature of the total space.

Of course one could dispense with the fibration entirely and just study the signature of local coefficient systems. This was done by Meyer who then went on to study in great depth the case of surface bundles over surfaces. He showed that every multiple of four occurs as the signature of a surface bundle over a surface, but did not give explicit topological examples. He gave an example of a local coefficient system with non-multiplicative signature over the surface of genus two.

Endo [End98] then went on to construct an example of a surface bundle over a surface with signature four. The genus of the base space of this example is 111, although the genus of the fibre is only three. To date no example of non-multiplicativity is known where the base space has a lower genus.

The multiplicativity of the signature has also been studied by Neumann [Neu78] who investigated which fundamental groups of the base space give rise to non-multiplicative signatures. He showed that the signature is always multiplicative for a large class of groups, including finite groups and free groups. The work of Karras, Kreck, Neumann and Ossa on the cutting and pasting of manifolds

[KKNO73] investigated the strong relationship of this subject with the multiplicativity of the signature, we do not go into detail here.

In [HKR05] (jointly with I. Hambleton and A. Ranicki) we proved that the signature of a fibre bundle of PL-manifolds is multiplicative modulo four. In this thesis we will extend this result to fibrations of Poincaré spaces.

1.2 The results concerning the signature of fibrations.

We will be concerned with fibrations $p : E \to B$ with particular properties. To save ourselves repeating the same conditions we introduce:

Assumption 1.2. *The space B is a connected finite CW-complex (the CW structure is part of the data) and the fibre $F = p^{-1}(b_0)$ has the homotopy type of a finite CW-complex. We have choices of base-points b_0 and e_0 such that $p(e_0) = b_0$.*

We say a fibration is *Poincaré* if F, E and B are Poincaré spaces and that the orientation of E is compatible with that of F and B.

Our two main results will be the following:

Theorem 7.2. *Let $F^m \to E^{n+m} \to B^n$ be a Poincaré fibration satisfying assumption 1.2, and such that $\tau(B) = 0 \in \mathrm{Wh}(\pi_1(B))$. Then*

$$\mathrm{sign}(E) = \mathrm{sign}(B)\mathrm{sign}(F) \pmod 4$$

Recall that for a Poincaré space X the Whitehead torsion $\tau(X) \in \mathrm{Wh}(\pi_1(X))$ is defined to be the Whitehead torsion of the Poincaré duality map $\tau(\phi_0 : C(\widetilde{X})^{n-*} \to C(\widetilde{X}))$. It is always zero if X is a manifold. We do not know whether this Theorem fails if the Whitehead torsion condition is not satisfied.

Theorem 8.1. *Let $F^{4m} \to E^{4n+4m} \to B^{4n}$ be a Poincaré fibration satisfying assumption 1.2, and such that the action of $\pi_1(B)$ on $(H_{2m}(F;\mathbf{Z})/torsion) \otimes \mathbf{Z}_2$ is trivial. Then*

$$\mathrm{sign}(E) = \mathrm{sign}(B)\mathrm{sign}(F) \pmod 8$$

S. Klaus and P. Teichner [KT03] have conjectured that this is true even if the dimension of B is not a multiple of four.

1.3 Symmetric L-theory.

This thesis will use Ranicki's chain complex techniques. We assume that the reader is familiar with chain complexes, chain maps, chain homotopies etc..2qw. We will present a more detailed description of the symmetric Poincaré complexes of Ranicki [Ran80a, Ran80b] in chapter 2; we will just give a brief description here. An n-dimensional symmetric complex (C, ϕ) consists of a chain complex C of modules over a ring with involution R, a chain equivalence $\phi_0 : C^{n-*} \to C$, a homotopy ϕ_1 between ϕ_0 and its dual, and higher homotopies ϕ_s. Such a complex is said to be Poincaré if ϕ_0 is a chain equivalence. If now (C, ϕ) is $4k$-dimensional and over the ring $\mathbf{Z}$ (or $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{C}$) then the signature $\operatorname{sign}(C, \phi)$ is defined to be the signature of the middle dimensional symmetric form $\phi_0 : H^{2k}(C) \to H_{2k}(C)$.

Let X be an n-dimensional Poincaré complex. Then we can form an n-dimensional symmetric complex $(C(\widetilde{X}), \phi)$ over the ring $\mathbf{Z}[\pi_1(X)]$ where $C(\widetilde{X})$ is a chain complex of X and the chain equivalence ϕ_0 represents cap product with the fundamental class:

$$\phi_0 = [X] \cap - : C(\widetilde{X})^{n-*} \to C(\widetilde{X})$$

We say that $(C(\widetilde{X}), \phi)$ *represents* $\widetilde{X}$.

The abelian group $L^n(R)$ consists of n-dimensional symmetric Poincaré complexes modulo the boundaries of $(n+1)$-dimensional symmetric complexes (see chapter 2) with composition given by direct sum. Ranicki [Ran80b] shows that the signature gives a well-defined map from $L^{4k}(\mathbf{Z})$ to $\mathbf{Z}$ and that moreover this map is an isomorphism.

We have a map $\epsilon : \mathbf{Z}[\pi] \to \mathbf{Z}$ of rings defined by $g \mapsto 1$ on the group elements and this induces a map of symmetric L-groups $L^{4k}(\mathbf{Z}[\pi]) \to L^{4k}(\mathbf{Z}) \cong \mathbf{Z}$. The image under this map of a symmetric Poincaré complex representing a Poincaré space X is the signature of the space X. The element $\sigma^*(\widetilde{X})$ of $L^n(\mathbf{Z}[\pi_1(X)])$ representing a space X is often referred to as the symmetric signature of $\widetilde{X}$.

Unfortunately not every element of $L^n(\mathbf{Z}[\pi])$ can be represented as a representative of a Poincaré space X. There exists a subtle variation on symmetric L-theory called *visible* symmetric L-theory, developed by Weiss [Wei92]. The groups $VL^n(\mathbf{Z}[\pi])$ have the advantage that every element may be geometrically realized; they are also easier to compute. There exists a map $VL^n(\mathbf{Z}[\pi]) \to L^n(\mathbf{Z}[\pi])$ by forgetting that the complexes are visible.

1.4 The algebraic surgery transfer map.

In many ways the theory we develop here is very much inspired by the algebraic surgery transfer map of Lück and Ranicki [LR92]. We will assume familiarity with the surgery theory of Wall [Wal99] and the algebraic theory of surgery of Ranicki [Ran80a, Ran80b]) in this section of the introduction. The surgery transfer is defined for a fibration $F \to E \to B$ where F is an m-dimensional manifold. It is a map:

$$p^! : L_n(\mathbf{Z}[\pi_1(B)]) \to L_{n+m}(\mathbf{Z}[\pi_1(E)])$$

which gives an algebraic description of the surgery transfer map of Quinn [Qui70]. This sends the surgery obstruction $\sigma_*(f, b)$ of an n-dimensional normal map $(f, b) : M \to X$ with reference map $X \to B$ to the surgery obstruction $\sigma_*(f^!, b^!)$ of the $(n+m)$-dimensional normal map $(f^!, b^!) : M^! \to X^!$ obtained by pullback from $p : E \to B$. It is a Theorem of Wall [Wal99] that every surgery obstruction in $L_n(\mathbf{Z}[\pi_1(B)])$ may be represented by a normal map $(f, b) : M \to X$ with reference map $X \to B$, so the geometric transfer is well-defined.

This is a very nice situation because the algebraic situation fits the geometric one precisely; this is to be expected because the quadratic L-groups can always be geometrically realized. One might hope to do something similar in the symmetric case, in other words construct a map algebraically:

$$p^! : L^n(\mathbf{Z}[\pi_1(B)]) \to L^{n+m}(\mathbf{Z}[\pi_1(E)])$$

however in the appendix to [LR92] Lück and Ranicki argued that such a map cannot be constructed using their algebraic techniques. It cannot even be constructed geometrically since not every element of $L^n(\mathbf{Z}[\pi_1(B)])$ can be realized as the symmetric signature of a Poincaré complex, even when n is large. There is a transfer map in the visible symmetric L-groups but this is rather unwieldy and we do not describe it here (see [Ran92]).

1.5 Describing the signature of a fibration algebraically.

One of the key steps in proving our main results will be to describe the signature of a fibration in terms of the symmetric complex of the base space and the action of $\pi_1(B)$ on the fibre. This can be seen as the algebraic analogue of Meyer's description of the signature of a fibration in terms of the intersection form on B with coefficients in a local coefficient system.

In chapter 4 we define a $(\mathbf{Z}, m)$-symmetric representation of a ring with involution R; this consists of a triple (A, α, U) where A is a free $\mathbf{Z}$-module, $\alpha : A^* \to A$ a map such that $\alpha^* = (-)^m \alpha$ and $U : R \to \mathrm{Hom}(A, A)^{op}$ a map of rings such that $u(r)^* \alpha = \alpha u(r^*)$. We can construct a $(\mathbf{Z}, m)$-symmetric representation (K, ϕ^F, U) of $\mathbf{Z}[\pi_1(B)]$ from a Poincaré fibration $F^{2m} \to E^{2m+2n} \to B^{2n}$ by defining $K = H_m(F; \mathbf{Z})/torsion$ and ϕ^F the form on K given by Poincaré duality. The map U is defined on pure group elements $g \in \mathbf{Z}[\pi_1(B)]$ by the action of g on $K = H_m(F; \mathbf{Z})/torsion$ and we extend this to the ring $\mathbf{Z}[\pi_1(B)]$ in the obvious way. This is very much analogous to the notion of a local coefficient system.

We can use the map U to form a twisted tensor product $M \otimes (A, \alpha, U)$ of a based left $\mathbf{Z}[\pi_1(B)]$-module with A as follows: We first identify M with $\mathbf{Z}[\pi_1(B)]^k$ using the basis and define

$$M \otimes (A, \alpha, U) = \bigoplus_k A$$

For morphisms $f : M \to N$ we define

$$f \otimes (A, \alpha, U) : M \otimes (A, \alpha, U) \to N \otimes (A, \alpha, U)$$

to be $U(f_{i,j})$ on the components $f_{i,j}$ of f with respect to the basis (in other words we consider f to be a matrix and apply U to every element). We can think of tensor product $- \otimes (A, \alpha, U)$ as a functor from the category of based left $\mathbf{Z}[\pi_1(B)]$ modules to the category of free $\mathbf{Z}$-modules. This functor induces a map:

$$- \otimes (A, \alpha, U) : L^n(\mathbf{Z}[\pi_1(B)]) \to L^{n+2m}(\mathbf{Z})$$

in symmetric L-theory described explicitly in chapter 4. We prove the following Theorem which allows us to study the signature of a fibration using chain complex techniques:

Theorem 4.9. *Let $F^{2m} \to E^{2n+2m} \to B^{2n}$ be a Poincaré fibration satisfying assumption 1.2, let $(C(\widetilde{B}), \phi^B)$ be a symmetric Poincaré complex representing B and denote by (K, ϕ^F, U) the $(\mathbf{Z}, m)$-symmetric representation constructed from the fibration. Then the signature of E is equal to the signature of the symmetric Poincaré complex $(C(\widetilde{B}), \phi^B) \otimes (K, \phi^F, U)$.*

We will refer to $(C(\widetilde{B}), \phi^B) \otimes (K, \phi^F, U)$ as the *twisted tensor product* of $(C(\widetilde{B}), \phi^B)$ and (K, ϕ^F, U). We will prove our two main theorem by proving the corresponding statement for visible symmetric complexes and $(\mathbf{Z}, m)$-symmetric representations and then by applying this theorem. Given this theorem one can naturally ask:

Question 1.3. *Is it possible (either geometrically or algebraically) to describe the symmetric signature $\sigma^*(E) \in L^{n+m}(\mathbf{Z}[\pi_1(E)])$ of the total space of a fibration in terms of the symmetric signature of the base space and the fibre and the action of $\pi_1(B)$ on the fibre?*

1.6 The chain complex of a fibration.

An important tool in this thesis will be a description of the chain complex of the total space of a fibration in terms of the base space and the action of $\pi_1(B)$ on the fibre; this is the subject of chapter 3. Obviously one cannot produce a complete description of $C(\widetilde{E})$ from this data, otherwise fibrations with simply connected base spaces wouldn't be very interesting. Instead we take advantage of the fact that the chain complex of the total space is a filtered complex and we describe the *associated complex* (defined in chapter 3) in these terms. This is very similar to using the Serre spectral sequence to study fibrations and many readers may at this point be wondering why we don't do so. The main disadvantage to working with spectral sequences is that taking homology produces some rather exotic modules when working over a group ring $\mathbf{Z}[\pi]$, and therefore it is rather unsuitable when one wants to use algebraic surgery theory. We can overcome this problem by working with chain complexes instead. The description of the chain complex of a fibration given here is quite general and may well be of independent interest. It is very much inspired by the work of Lück [Lüc86] on the transfer map in algebraic K-theory. The key theorem (Theorem 3.11) describes not only the chain complex of the fibration but also the diagonal approximation. It is the link between topology and algebra which we require. We will postpone the statement of this Theorem until chapter 3 because even this requires some rather technical language that would be out of place here.

1.7 Absolute Whitehead torsion.

We develop a theory of absolute Whitehead torsion which refines the usual theory. This can be read quite separately from the rest of the thesis (indeed this chapter is essentially the preprint [Kor05]) and is of sufficient independent interest that the author feels that its intrusion in the title is entirely justified. The motivation for introducing this theory is that it may be used to detect the signature of a symmetric Poincaré complex modulo four, generalizing the formula of Hirzebruch and Koh [HNK71], that for a unimodular symmetric form ϕ over the integers

$$\mathrm{sign}(\phi) = \det(\phi) + \mathrm{rank}(\phi) - 1 \pmod 4$$

Another motivation is that the definition used in [HRT87] of the absolute torsion of a round ($\chi(C) = 0$) symmetric Poincaré complex does not have the desired properties.

The Whitehead torsion of a homotopy equivalence $f : X \to Y$ of finite CW complexes is an element of the Whitehead group of $\pi = \pi_1(X) = \pi_1(Y)$

$$\tau(f) \;=\; \tau(\widetilde{f} : C(\widetilde{X}) \to C(\widetilde{Y})) \in Wh(\pi) \;=\; K_1(\mathbf{Z}[\pi])/\{\pm\pi\}\ ,$$

with $\widetilde{f}$ the induced chain equivalence of based f.g. free cellular $\mathbb{Z}[\pi]$-module chain complexes (see e.g [Mil66]). The Whitehead torsion of a finite n-dimensional Poincaré complex X is

$$\tau(X) \;=\; \tau([X]\cap - : C(\widetilde{X})^{n-*} \to C(\widetilde{X})) \in Wh(\pi)\ .$$

In this chapter we extend the methods of [Ran85] to consider absolute Whitehead torsion invariants for homotopy equivalences of certain finite CW complexes and finite Poincaré complexes, which take values in $K_1(\mathbb{Z}[\pi])$ rather than $Wh(\pi)$.

The absolute torsion of a finite contractible chain complex of finitely generated based R-modules C is defined by

$$\tau(C) \;=\; \tau(d + \Gamma : C_{odd} \to C_{even}) \in K_1(R)\ .$$

for a chain contraction Γ; it is independent of the choice of Γ. The algebraic mapping cone of a chain equivalence of finite chain complexes of finitely generated based R-modules $f : C \to D$ is a contractible chain complex $\mathcal{C}(f)$. The naive absolute torsion $\tau(\mathcal{C}(f)) \in K_1(R)$ only has good additive and composition formulae modulo $\mathrm{Im}(K_1(\mathbb{Z}) \to K_1(R))$. Likewise, the naive definition of the torsion of an n-dimensional symmetric Poincaré complex (C, ϕ)

$$\tau(C,\phi) \;=\; \tau(\mathcal{C}(\phi_0 : C^{n-*} \to C))$$

only has good cobordism and additivity properties in $\widetilde{K}_1(R)$. The Tate $\mathbf{Z}_2$-cohomology class

$$\tau(C,\phi) \in \widehat{H}^n(\mathbf{Z}_2; K_1(R))$$

may not be defined, and even if defined may not be a cobordism invariant.

In [Ran85] Ranicki developed a theory of absolute torsion for chain equivalences of round chain complexes, that is chain complexes C satisfying $\chi(C) = 0$. This absolute torsion has a good composition formula but it is not additive, and for round Poincaré complexes $\tau(C, \phi_0)$ is not a cobordism invariant (contrary to the assertions of [Ran89, 7.21, 7.22]).

There are two main aims of this chapter, firstly to develop a more satisfactory definition of the absolute torsion of a chain equivalence with good additive and composition formulae and secondly to define an absolute torsion invariant of Poincaré complexes which behaves predictably under cobordism. Section 5.1 is devoted to the first of these aims. Following [Ran85] we work in the more general context of an additive category $\mathbb{A}$. The chief novelty here is the introduction of a *signed* chain complex; this is a pair (C, η_C) where C is a finite chain complex and η_C is a "sign" term living in $K_1(\mathbb{A})$, which will be made precise in section 5.1. We give definitions for the sum and suspension of two signed chain complexes, and we define the absolute torsion of a chain equivalence of signed chain complexes.

$$\tau^{NEW}(f : C \to D) \in K_1^{iso}(\mathbb{A})$$

This gives us a definition of absolute torsion with good additive and composition formulae at the cost of making the definition more complicated by adding sign terms to the chain complexes. This definition is similar to the one given in [Ran85], indeed if the chain complexes C and D are round and $\eta_C = \eta_D = 0$ then the definition of the absolute torsion of a chain equivalence $f : C \to D$ is precisely that given in [Ran85]. When working over a ring R the absolute torsion defined here reduces to the usual torsion in $\widetilde{K_1}(R)$.

In section 5.3 we work over a category with involution and define the dual of a signed chain complex. We can then define in section 5.5 the absolute torsion of a symmetric Poincaré complex to be the absolute torsion of the chain equivalence $\phi_0 : C^{n-*} \to C$. This new invariant is shown to be additive and to have good behavior under *round* algebraic cobordism. Although we have to choose a sign η_C in order to define the absolute torsion of ϕ_0, we show that the absolute torsion is independent of this choice.

In chapter 6 we compute the absolute torsion of fibration using the techniques of [HKR05]. This is a generalization of the result of [HKR05] to fibrations of Poincaré spaces. In chapter 7 we compute the absolute torsion of a twisted tensor product in order to prove that the signature of a fibration is multiplicative modulo four, one of our two main Theorems.

1.8 The generalized Pontrjagin square.

To investigate the multiplicativity of the signature modulo eight we will make use of the fact that the signature modulo eight can be determined from the Pontrjagin square (see chapter 8 for details). The Pontrjagin square is a cohomology operation:

$$\mathcal{P} : H^k(X; \mathbf{Z}_2) \to H^{2k}(X; \mathbf{Z}_4)$$

which refines the $\mathbf{Z}_2$-valued cup product; i.e. $i_*\mathcal{P}(x) = x \cup x \in H^{2k}(X;\mathbf{Z}_2)$ where $i_* : H^{2k}(X;\mathbf{Z}_4) \to H^{2k}(X;\mathbf{Z}_2)$ is the map induced by the non-trivial map $\mathbf{Z}_4 \to \mathbf{Z}_2$. If X is a $2k$-dimensional Poincaré space then we have a map:

$$\begin{aligned} \mathcal{P} : H^k(X;\mathbf{Z}_2) &\to \mathbf{Z}_4 \\ x &\mapsto \langle [X], \mathcal{P}(x) \rangle \end{aligned}$$

given by the Pontrjagin square evaluated on the fundamental class. One can also define a map $\mathcal{P} : H^k(C;\mathbf{Z}_2) \to \mathbf{Z}_4$ for $2k$-dimensional symmetric Poincaré complexes (C,ϕ) over $\mathbf{Z}$ which is consistent with that defined for spaces. It is a Theorem of Morita ([Mor71], see also Theorem 8.5) that in the case where k is even the Arf invariant of $\mathcal{P}$ detects the signature of a symmetric Poincaré space modulo eight. Our novelty is to introduce a generalized Pontrjagin square for a $2k$-dimensional symmetric Poincaré complex over a ring with involution R and an ideal I in the ring:

$$\mathcal{P}_{(C,\phi),I} : H^k(C;R/I) \to R/(I^2 + 2I)$$

which is a further refinement of the Pontrjagin square. If we now take R to be a group ring $\mathbf{Z}[\pi]$ and the ideal $I = \epsilon^{-1}(2\mathbf{Z})$ we have a map:

$$\mathcal{P}_{(C,\phi),I} : H^k(C;\mathbf{Z}_2) \to \mathbf{Z}[\pi]/I^2 = \mathbf{Z}_4 \oplus H_1(\pi;\mathbf{Z}_2) \tag{1.8.1}$$

We say a $(\mathbf{Z},m)$-symmetric representation is $\mathbf{Z}_2$-trivial if

$$U(r) \otimes \mathbf{Z}_2 = \epsilon(r) I \otimes \mathbf{Z}_2 : A \otimes \mathbf{Z}_2 \to A \otimes \mathbf{Z}_2$$

This is satisfied for the $(\mathbf{Z},m)$-symmetric representation of a fibration $F^{2m} \to E^{2m+2n} \to B^{2n}$ if the action of $\pi_1(B)$ on $(H_m(F;\mathbf{Z})/torsion) \otimes \mathbf{Z}_2$ is trivial.

It turns out that the Pontrjagin square on a twisted tensor product of a symmetric complex (C,ϕ) and a $\mathbf{Z}_2$-trivial $(\mathbf{Z},m)$-symmetric representation (A,α,U) can be expressed in terms of the map 1.8.1 on (C,ϕ) and the map α. Moreover it is seen to depend only on the diagonal elements of the matrices $U(g)$. This implies that the signature of the twisted product is equal to the sum of the signatures of twisted products of (C,ϕ) and rank 1 $(\mathbf{Z},m)$-symmetric representations. Therefore it is sufficient to demonstrate modulo eight multiplicativity of the signature for such representations.

1.9 Chapter outlines.

In chapter 2 we recall all of the necessary theory of symmetric Poincaré complexes. In chapter 3 we derive our algebraic description of a fibration; this is used in chapter 4 to prove Theorem 4.9 which gives us an algebraic description of the signature of a fibration. We develop the theory of Absolute Whitehead torsion in chapter 5 which we apply in chapter 6 to compute the absolute torsion of the total space of a fibration and in chapter 7 to show that the signature of a fibration is multiplicative modulo four. Finally in chapter 8 we prove our modulo eight result.

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Chapter 2

Chain complexes, the algebraic theory of surgery and the symmetric construction.

In this chapter we will recall some of the algebraic theory of Ranicki [Ran80a, Ran80b], which underpins the results of this thesis. As the name suggests, the Algebraic Theory of Surgery was origionally developed to give a purely algebraic formulation of the surgery theory of Wall [Wal99] via *quadratic* complexes. We will not be directly concerned with surgery theory here, although we will be comparing our results with the surgery transfer of Lück and Ranicki [LR92]. The main algebraic basis of this thesis will be the closely related *symmetric* complexes which, in the same way that quadratic complexes are an algebraic model of surgery obstructions, provide an algebraic model of certain properties of a space which arise from the diagonal map. The process of extracting this algebra from an actual topological space is called the symmetric construction, originally due to Mischenko but in the present formulation to Ranicki [Ran80b].

2.1 Symmetric L-theory

All chain complexes are finite and non-zero only in positive dimensions unless stated otherwise.

We will develop the theory in the most general way with chain complexes in an additive category $\mathbb{A}$. We will write $\mathbb{A}(R)$ for the additive category of finitely generated free left R-modules and $\mathbb{A}^{based}(R)$ if in addition the modules are to be based. We denote by $\mathbb{D}(\mathbb{A})$ the additive category of finite chain complexes in an additive category $\mathbb{A}$ with morphisms the chain homotopy classes of chain maps. This category is referred to throughout as the derived category . We write $\mathbb{D}(R)$ as shorthand for $\mathbb{D}(\mathbb{A}(R))$.

Following [Ran89] we define an *involution* on an additive category $\mathbb{A}$ to be a contravariant functor

$$* : \mathbb{A} \to \mathbb{A};\ M \to M^*,\ (f : M \to N) \to (f^* : N^* \to M^*)$$

together with a natural equivalence

$$e : id_{\mathbb{A}} \to ** : \mathbb{A} \to \mathbb{A};\ M \to (e(M) : M \to M^{**})$$

such that for any object M of $\mathbb{A}$

$$e(M^*) = (e(M)^{-1})^* : M^* \to M^{***}$$

A functor of derived categories with involution consists of a functor $F : \mathbb{A} \to \mathbb{B}$ of the underlying categories together with a natural equivalence $H : F* \to *F : \mathbb{A} \to \mathbb{B}$ such that for every $M \in \mathbb{A}$ there is a commutative diagram:

$$\begin{tikzcd} F(M) \arrow[r, "e(F(M))"] \arrow[d, "F(e(M))"'] & F(M)^{**} \arrow[d, "G(M)^*"] \\ F(M^{**}) \arrow[r, "G(M^*)"] & F(M^*)^* \end{tikzcd}$$

Given a ring with involution R the category $\mathbb{A}(R)$ and $\mathbb{A}^{based}(R)$ are categories with involution $*M = M^*$. We the left action of an element $r \in R$ on M^* is defined to be the right action of the involution applied to r. When we work over a ring with involution we have left actions as well as right actions so the tensor product $M \otimes_R N$ is always well-defined.

We write $SC_r = C_{r-1}$ for the suspension of a chain complex. We should state the sign conventions that we'll be using:

Sign Convention 2.1. *The tensor product $C \otimes_S D$ of chain complexes C and D of right, respectively left S-modules is a chain complex given by:*

$$(C \otimes D)_n = \bigoplus_{i+j=r} C_i \otimes D_j$$

$$d_{C\otimes D}(x \otimes y) = x \otimes d_D(y) + (-)^j d_C(x) \otimes y$$

Sign Convention 2.2. *We define the algebraic mapping cone of a chain map $f : C \to D$ to be the chain complex $\mathcal{C}(F)$ given by :*

$$d_{\mathcal{C}(f)} = \begin{pmatrix} d_D & (-)^{r+1} f \\ 0 & d_C \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \to \mathcal{C}(f)_{r-1} = D_{r-1} \oplus C_{r-2}$$

Sign Convention 2.3. *Given a n-dimensional chain complex*

$$C \,:\, C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \xrightarrow{d} \ldots \xrightarrow{d} C_0$$

we use the following sign convention for the dual complex C^{n-*}.

$$d_{C^{n-*}} = (-)^r d_C^* \,:\, C^{n-r} \to C^{n-r+1}$$

We define the duality isomorphism T as:

$$T \,:\, \mathrm{Hom}_{\mathbb{A}}(C^p, D_q) \to \mathrm{Hom}_{\mathbb{A}}(D^q, C_p)\,;\, \phi \to (-)^{pq}\phi^*$$

Before we get into the full technical detail of symmetric complexes we start off with a slightly easier notion that will be useful later on. We define an *n-dimensional symmetric chain equivalence* (C, ϕ) in $\mathbb{A}$ to be an n-dimensional chain complex C in $\mathbb{A}$ and a chain equivalence $\phi_0 : C^{n-*} \to C$ such that $T\phi_0 : C^{n-*} \to C$ is chain equivalent to ϕ_0. We see that ϕ induces an isomorphism in homology $\phi : H^{n-r}(C) \to H_r(C)$ such that $\phi = (-)^{r(n-r)}\phi^* : H^{n-r}(C) \to H_r(C)$. We define the *signature* of a $4k$-dimensional symmetric chain equivalence in $\mathbb{A}(\mathbf{Z})$ (or $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{C}$) to be the signature of the form $\phi : H^{2k}(C) \to H_{2k}(C)$. We denote this by $\mathrm{sign}(C, \phi)$.

Definition 2.4.

1. *An n-*dimensional symmetric complex *(C, ϕ_0) is a finite chain complex C in $\mathbb{A}$, together with a collection of morphisms*

   $$\phi = \{\phi_s \colon C^{n-r+s} \to C_r \mid s \geq 0\}$$

   such that

   $$d_C\phi_s + (-)^r\phi_s d_C^* + (-)^{n+s+1}(\phi_{s-1} + (-)^s T\phi_{s-1}) = 0$$
   $$: C^{n-r+s-1} \to C_r \quad (s \geq 0, \phi_{-1} = 0)$$

   Hence $\phi_0 : C^{n-*}$ is a chain map and ϕ_1 is a chain homotopy $\phi_1 : \phi_0 \simeq T\phi_0$.

2. *The complex is said to be* Poincaré *if ϕ_0 is a chain equivalence. In this case (C, ϕ_0) is an n-dimensional symmetric chain equivalence.*

3. *A* morphism *between n-dimensional symmetric complexes (C, ϕ) and (C', ϕ') consists of a chain map $f : C \to C'$ and morphisms $\sigma_s : C'^{n+1+s-r} \to C'_r$ $s \geq 0$ such that*

   $$\phi'_s - f\phi_s f^* = d_C\sigma_s + (-)^r\sigma_s d_C^* + (-)^{n+s}(\sigma_{s-1} + (-)^s T\sigma_{s-1}) : C^{n-r+s} \to C_r$$

   (in particular $\phi'_0 \simeq f\phi_0 f^*$). Such a morphism is said to be a homotopy equivalence *if f is a chain equivalence.*

4. *The* boundary $(\partial C, \partial\phi)$ *of a n-dimensional symmetric complex (C,ϕ) is the $(n-1)$-dimensional symmetric Poincaré complex defined by*

$$d_{\partial C} = \begin{pmatrix} d_C & (-)^r\phi_0 \\ 0 & (-)^r d_C^* \end{pmatrix} : \partial C_r = C_{r+1} \oplus C^{n-r} \to$$
$$\partial C_{r-1} = C_r \oplus C^{n+1-r}$$
$$\partial\phi_0 = \begin{pmatrix} (-)^{n-r-1}T\phi_1 & (-)^{rn} \\ 1 & 0 \end{pmatrix} : \partial C^{n-r-1} = C^{n-r} \oplus C_{r+1} \to$$
$$\partial C_r = C_{r+1} \oplus C^{n-r}$$
$$\partial\phi_s = \begin{pmatrix} (-)^{n-r-1}T\phi_{s+1} & (-)^{rn} \\ 0 & 0 \end{pmatrix} : \partial C^{n-r+s-1} = C^{n+s-r} \oplus C_{r-s+1}$$
$$\to \partial C_r = C_{r+1} \oplus C^{n-r}$$

We will show how symmetric complexes occur topologically in the next section. A more intrinsic way of defining symmetric complexes in $\mathbb{A}(R)$ (R a ring with involution) is as follows: We have a slant product:

$$\backslash : C \otimes_S D \to \mathrm{Hom}(C^{-*}, D)$$
$$x \otimes y \mapsto (f \mapsto f(\bar{x}).y)$$

Let W be the standard free resolution of $\mathbf{Z}$ over $\mathbf{Z}[\mathbf{Z}_2]$:

$$W \ : \ \ldots \to \mathbf{Z}[\mathbf{Z}_2] \xrightarrow{1-T} \mathbf{Z}[\mathbf{Z}_2] \xrightarrow{1+T} \mathbf{Z}[\mathbf{Z}_2] \xrightarrow{1-T} \mathbf{Z}[\mathbf{Z}_2]$$

Then for a finite chain complex C we may regard a symmetric complex (C,ϕ) to be an n-cycle in the chain complex

$$\phi \in \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C \otimes_R C)$$

(the action of $\mathbf{Z}_2$ on $C \otimes_R C$ is given by interchanging the components) with the morphisms ϕ_i being the images under ϕ of the canonical generators of W after applying the slant product.

Definition 2.5. *We define an $(n+1)$-dimensional symmetric pair $(f : C \to D, (\delta\phi, \phi))$ to be a symmetric complex (C,ϕ), a chain map $f : C \to D$ and $\{\delta\phi_s : D^{n+1+s-r} \to D_r\}$ a collection of morphisms such that:*

$$(-)^{n+1} f\phi_s f^* = d_D\delta\phi_s + (-)^r\delta\phi_s d_D^* + (-)^{n+s}(\delta\phi_{s-1} + (-)^s T\delta\phi_{s-1})$$
$$: D^{n-r+s} \to D_r \quad (s \geq 0, \delta\phi_{-1} = 0)$$

The pair is said to be Poincaré *if the chain map:*

$$\begin{pmatrix} \delta\phi_0 \\ (-)^{n+r+1}\phi_0 f^* \end{pmatrix} : D^{n+1-*} \to \mathcal{C}(f)$$

is a chain equivalence.

The condition of a pair being Poincaré is equivalent to saying that the chain map:

$$\begin{pmatrix} \delta\phi_0 & f\phi_0 \end{pmatrix} : \mathcal{C}(f)^{n+1-*} \to D$$

is a chain equivalence. The reason we bring this up in the first place is that there is a Theorem of Ranicki [Ran80a] which states that a symmetric Poincaré complex (C, ϕ) is homotopy equivalent to the boundary of some symmetric complex if and only if there exists a symmetric Poincaré pair $(f : C \to D, (\delta\phi, \phi))$. We say that a symmetric Poincaré complex satisfying these conditions is *null-cobordant.* We often refer to a pair as a *null-cobordism* of (C, ϕ).

We now define the *symmetric L-groups* $L^n(\mathbb{A})$ of an additive category with involution to be the set of n-dimensional symmetric Poincaré complexes (with addition given by direct sum) modulo those which are null-cobordant. In particular this implies that homotopy equivalent symmetric Poincaré complexes represent the same element in $L^n(\mathbb{A})$. We will write $L^n(R) = L^n(\mathbb{A}(R))$. As an example we recall from [Ran80b] the computation of the symmetric L-groups of $\mathbf{Z}$.

Proposition 2.6. *The symmetric L-groups of* $\mathbf{Z}$ *for* $n \geq 0$ *are as follows:*

$$L^n(\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{(signature)} \\ \mathbf{Z}_2 & \text{(de Rham invariant)} \\ 0 & \\ 0 & \end{cases} \quad \text{for } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases}$$

There is a lot more that can be said about symmetric L-theory and how it relates to the other constructions of algebraic surgery; for more information see [Ran80a, Ran80b].

2.2 The symmetric construction.

We now explain how the above algebra occurs topologically. We will show that for an n-dimensional Poincaré space X there is an associated symmetric Poincaré complex over $\mathbf{Z}[\pi_1(X)]$.

Let X be a space with the homotopy type of a CW-complex. We may as well replace X by such a complex so we can form the chain complex $C(X)$. Let $\widetilde{X} \to X$ be the universal cover of X. Then the chain complex $C(\widetilde{X})$ may be regarded as a chain complex of finite free left $\mathbf{Z}[\pi_1(X)]$-modules. For any space we have the diagonal map $\widetilde{X} \to \widetilde{X} \times \widetilde{X}$; $x \mapsto (x, x)$. If we approximate this via the G-CW-approximation Thorem we get a chain map:

$$\Delta^X : C(\widetilde{X}) \to C(\widetilde{X}) \otimes_{\mathbf{Z}} C(\widetilde{X})$$

which extends (via acyclic model theory, see e.g. [MT68], [Ran80b]) to a map:

$$\Delta^X : C(\widetilde{X}) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(\widetilde{X}) \otimes_{\mathbf{Z}} C(\widetilde{X}))$$

We can now tensor with $\mathbf{Z}$ to get a map:

$$\Delta^X \otimes_{\mathbf{Z}[\pi_1(X)]} \mathbf{Z} : C(X) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(\widetilde{X}) \otimes_{\mathbf{Z}[\pi_1(X)]} C(\widetilde{X}))$$

If now X is an n-dimensional Poincaré space then we can find a chain level representative of the orientation class $[X]$ on the left hand side. The image of $[X]$ under the map $\Delta^X \otimes_{\mathbf{Z}[\pi_1(X)]} \mathbf{Z}$ determines a symmetric Poincaré complex $(C(\widetilde{X}), \phi)$. We say that this symmetric Poincaré complex *represents* $\widetilde{X}$. The process by which $(C(\widetilde{X}), \phi)$ is obtained from $\widetilde{X}$ is often referred to as the *symmetric construction.* Note that the homotopy class of $(C(\widetilde{X}), \phi)$ is independent of any choices that may have been made. We write $\sigma^*(\widetilde{X}) \in L^n(\mathbf{Z}[\pi_1(X)])$ for the class of $(C(\widetilde{X}), \phi)$ in the symmetric L-group and refer to this as the *symmetric signature* of X.

We don't have to take the universal cover of X as our starting point. For every regular covering $\hat{X} \to X$ with group of transformations π we can play the same game and obtain an element of $L^n(\mathbf{Z}[\pi])$. However this will not contain any more information than the symmetric Poincaré complex obtained from the universal cover. For example, let $X \to Y$ be a map with X a Poincaré space and let $\hat{X}$ be the pull-back of the universal cover of Y. Then the symmetric Poincaré complex representing $\hat{X}$ can be obtained from $(C(\widetilde{X}), \phi)$ by tensoring with $\mathbf{Z}[\pi_1(Y)]$ thus:

$$(C(\widetilde{X}) \otimes_{\mathbf{Z}[\pi_1(X)]} \mathbf{Z}[\pi_1(Y)], \phi \otimes_{\mathbf{Z}[\pi_1(X)]} \mathbf{Z}[\pi_1(Y)])$$

Here the action of $\mathbf{Z}[\pi_1(X)]$ on $\mathbf{Z}[\pi_1(Y)]$ is determined by the induced map $\pi_1(X) \to \pi_1(Y)$. In particular by tensoring with $\mathbf{Z}$ we can obtain an element of $L^n(\mathbf{Z})$ representing the trivial cover. If $n = 4k$ then the signature of $\sigma^*(X) \in L^{4k}(\mathbf{Z})$ is just the signature of X.

2.3 Visible symmetric complexes

The symmetric L-groups have the disadvantage that not every element of $L^n(\mathbf{Z}[\pi])$ is realizable as the symmetric signature of a Poincaré space $\sigma^*(\widetilde{X})$. However Michael Weiss has developed a variation on symmetric L-theory called visible symmetric L-theory [Wei92]. He constructs groups $VL^n(\mathbf{Z}[\pi])$ in which every element can be represented by a symmetric Poincaré complex (for high enough dimension). We will discuss only group rings over the integers, see [Wei92] for a complete description.

Let $\mathbf{Z}[\pi]$ be a group ring and let P be a right free resolution of $\mathbf{Z}$ over $\mathbf{Z}[\pi]$. Then a visible symmetric complex consists of a chain complex C in $\mathbb{A}(\mathbf{Z}[\pi])$ along with an n-cycle in:

$$\phi \in P \otimes_{\mathbf{Z}[\pi]} \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C \otimes_{\mathbf{Z}} C)$$

The augmentation map $P \to \mathbf{Z}$ determines a natural map:

$$P \otimes_{\mathbf{Z}[\pi]} \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C \otimes_{\mathbf{Z}} C) \to \mathrm{Hom}(W, C \otimes_{\mathbf{Z}[\pi]} C)$$

so we have a natural way of making a symmetric complex from a visible symmetric one. A visible symmetric complex is said to be *Poincaré* if it is Poincaré as a symmetric complex. We write $VL^n(\mathbf{Z}[\pi])$ for the group of visible Poincaré complexes modulo the boundaries of such complexes. There is a natural map:

$$VL^n(\mathbf{Z}[\pi]) \to L^n(\mathbf{Z}[\pi])$$

We will only really be concerned with symmetric complexes that occur as visible symmetric complexes so we're not going to be saying any more about $VL^n(\mathbf{Z}[\pi])$. However it should be noted that it is (in principle) easier to compute than $L^n(\mathbf{Z}[\pi])$. To get the right feel for visible symmetric complexes the reader should try and prove this lemma (quoted from [Ran92]).

Lemma 2.7. *Let (C, ϕ) be a 0-dimensional visible symmetric complex over $\mathbf{Z}[\pi]$. Then for all $x \in C_0$:*

$$\langle \phi_0(x), x \rangle$$

is of the form $a + b + b^*$, with $a \in \mathbf{Z}$ and $b \in \mathbf{Z}[\pi]$

A symmetric Poincaré complex which occurs as the symmetric signature of a Poincaré space X is always visible (see [Wei92] page 466 for details).

Chapter 3

The chain complex of a fibration.

All topological spaces will be compactly generated.

The aim of this chapter is to prove a theorem linking algebra and topology (Theorem 3.11). This theorem extracts just enough information from the topological situation of a fibration $p : E \to B$ satisfying assumption 1.2 to allow us to apply the algebraic techniques developed later in this thesis. This is the only chapter of this thesis with a significant amount of "hands on" topology in it and as such can be read quite independently of the rest of this thesis. Similarly there is nothing after the statement of Theorem 3.11 which the rest of the thesis depends on.

The theory is very much inspired by [HKR05],[LR88] and especially [Lüc86], indeed much of the notation comes from here and a few of the arguments. We always note in the text where this is the case. The theory developed here differs from [Lüc86] in two significant aspects:

- In [Lüc86] Lück was concerned solely with the algebraic K-theory transfer map $p^! : K_i(\mathbf{Z}[\pi_1(B)]) \to K_i(\mathbf{Z}[\pi_1(E)])$ $(i = 0, 1)$ and was able to take advantage of the existence of a geometric realization of the elements of $\mathrm{Wh}(\pi_1(B))$. In other words he could make the geometry match the algebra whereas we will have to make the algebra match the geometry. Specifically we will have to algebraically model filtered complexes to develop a filtered version of this theory (see the section on filtered complexes below).

- Since we will be concerned with Poincaré duality we will have to consider the diagonal approximation on the total space.

We should also mention the relationship between this theory and that developed in [HKR05] for PL-manifolds. The definition of a CW-model 3.14 is a reflection of that of a "pointed fibre bundle torsion structure" in [HKR05], and is also followed by a "realization" theorem (Theorem 3.15). The advantage to our approach is not only that it holds for more general topological spaces but that we avoid having to deal with absolute torsion in the topological part of the argument. This not only cleans it up but also allows the theory to be applied in other situations.

3.1 Filtered complexes and filtered spaces.

3.1.1 Filtered complexes.

We recall the following definition from [HKR05]. Let $\mathbb{A}$ be an additive category.

Definition 3.1. *1. A k-*filtered object *F_*M in $\mathbb{A}$ is an object M in $\mathbb{A}$ together with a direct sum decomposition:*

$$M = M_0 \oplus M_1 \oplus \ldots \oplus M_k$$

which we regard as a length k filtration

$$F_{-1}M = 0 \subset F_0M \subset F_1M \subset \ldots \subset F_kM = M$$

with

$$F_jM = M_0 \oplus M_1 \oplus \ldots \oplus M_j \quad (0 \leq j \leq k).$$

We refer to M_r as the r-th filtration quotient *of M.*

2. A filtered morphism *$f : F_*M \to F_*N$ of k-filtered objects in $\mathbb{A}$ is a morphism in $\mathbb{A}$ of the type*

$$f = \begin{pmatrix} f_0 & f_1 & f_2 & \cdots & f_k \\ 0 & f_0 & f_1 & \cdots & f_{k-1} \\ 0 & 0 & f_0 & \cdots & f_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f_0 \end{pmatrix} : M = \bigoplus_{s=0}^{k} M_s \to N = \bigoplus_{s=0}^{k} N_s$$

so that

$$f(F_jM) \subseteq F_jN \quad (0 \leqslant j \leqslant k)\ .$$

The (u, v)-component of this upper triangular matrix is a morphism $f_{v-u} \colon M_v \to N_u$, $0 \leqslant u \leqslant v \leqslant k$, where $f_j \colon M_* \to N_{*-j}$, $0 \leqslant j \leqslant k$, are graded morphisms in $\mathbb{A}$. We refer to the f_j as the component morphisms *of f.*

3. We denote by $\mathrm{Fil}_k(\mathbb{A})$ the additive category whose objects are filtered objects in $\mathbb{A}$ and whose morphisms are filtered morphisms.

4. A filtered complex *F_*C in $\mathbb{A}$ is a chain complex in $\mathrm{Fil}_k(\mathbb{A})$. We write C_r for the r-th object in the chain complex and write $C_{r,s}$ for the s-th filtration quotient of C_r.*

5. *A* filtered map $f : F_*C \to F_*D$ *of filtered complexes is a chain map in* $\mathrm{Fil}_k(\mathbb{A})$.

Example 3.2. *Let C and D be chain complexes in $\mathbb{A}(R)$ and $\mathbb{A}(R)$ respectively, where R and S are rings. Then the tensor product chain complex $C \otimes_{\mathbf{Z}} D$ may be regarded as a filtered complex with $(C \otimes D)_{r,s} = C_s \otimes D_{r-s}$.*

3.1.2 The associated complex

We will now define the associated complex of a filtered complex. This is a chain complex in the derived category $\mathbb{D}(\mathbb{A})$, in other words it is a chain complex whose objects are chain complexes in $\mathbb{A}$ and whose differentials are chain homotopy classes of chain maps.

Definition 3.3. *Let F_*C be a k-filtered complex in $\mathbb{A}$. We write $d_j : C_{r,s} \to C_{r-1,s-j}$ for the component morphisms of the differential d. The* associated complex *$G_*(C)$ is a k-dimensional complex in $\mathbb{D}(\mathbb{A})$*

$$G_*(C) \; : \; G_k(C) \to \ldots \to G_r(C) \xrightarrow{G_r d} G_{r-1}(C) \to \ldots \to G_0(C)$$

The objects $G_r(C)$ are chain complexes in $\mathbb{A}$ given by:

$$d_{G_r C} = d_0 \colon G_r(C)_s = C_{r+s,r} \to G_r(C)_{s-1} = C_{r+s-1,r}$$

and the differentials G_*d are the chain homotopy classes of the maps

$$G_r d = (-)^s d_1 \colon G_r(C)_s = C_{r+s,r} \to G_{r-1}(C)_s = C_{r+s-1,r-1}.$$

Example 3.4. *The associated complex of a tensor product $C \otimes D$ with the filtered structure defined in example 3.2 has objects:*

$$G_r(C \otimes D) = C_r \otimes D$$

and differentials

$$G_*d = d^C \otimes 1 : C_r \otimes D \to C_{r-1} \otimes D$$

3.1.3 Filtered spaces.

Here we define the notion of a filtered space and prove a two technical lemmas which will be useful later on. We work in the category of compactly generated spaces.

Definition 3.5. • *A* k-filtered space X *is a topological space* X *together with a series of sub-spaces*

$$X_{-1} = 0 \subset X_0 \subset X_1 ... \subset X_k = X$$

To avoid introducing extra terminology we also insist that each inclusion $X_j \subset X_{j+1}$ *be a cofibration - this will always be the case in this thesis. We will sometimes have a base-point* $x_0 \in X_0$.

- *A* filtered map *between two filtered spaces* X *and* Y *is a map* $f : X \to Y$ *such that* $f(X_j) \subset Y_j$. *A* filtered homotopy *between two such maps* f *and* f' *is a homotopy* $H : X \times I \to Y$ *such that* $H(X_j \times I) \subset Y_j$. *A* filtered homotopy equivalence *is a filtered map* $f : X \to Y$ *such that there exists a filtered map* $g : Y \to X$ *such that* fg *and* gf *are filtered homotopic to the identity maps on* Y *and* X *respectively.*

- *A* k-filtered CW-complex X *is a CW-complex* X *together with a series of sub-complexes*

$$X_{-1} = 0 \subset X_0 \subset ... \subset X_k = X$$

The cellular chain complex $C(X)$ *is filtered with*

$$F_jC(X) = C(X_j)$$

A cellular map $f : X \to Y$ *between filtered CW-complexes* X *and* Y *is said to be* filtered *if* $f(X_j) \subset Y_j$. *In this case the chain map* $f_* : C(W) \to C(Y)$ *is filtered.*

Notice that $G_jC(X) = S^{-j}C(X_j, X_{j-1})$ for a filtered CW-complex X.

Our main example of a filtered complex arises from a fibration $p : E \to B$ satisfying assumption 1.2. The CW-complex B is filtered by its skeleta and we give E a filtered structure by defining $E_k := p^{-1}(B_k)$. Each inclusion $E_{k-1} \subset E_k$ is a cofibration because we're working in the category of compactly generated spaces.

Lemma 3.6. *A filtered map* $f : X \to Y$ *is a filtered homotopy equivalence if and only if each* $f_j : X_j \to Y_j$ *is a homotopy equivalence of unfiltered spaces.*

Proof. Apply the result of Brown [Bro68] 7.4.1 inductively. $\square$

A filtered cellular map $f : X \to Y$ of filtered CW-complexes induces a filtered map $F_*f : F_*C(X) \to F_*C(Y)$. The following lemma should help the reader get a feel for how the algebra matches the topology.

Lemma 3.7. *Let f, f' be filtered homotopic filtered cellular maps between cellular maps X and Y. Then:*

$$G_*f = G_*f' : C(X) \to C(Y)$$

Proof. The map $f_j : G_jC(X) \to C(Y)$ depends only on the homotopy class of the map of pairs $(f_j, f_{j-1}) : (X_j, X_{j-1}) \to (Y_j, Y_{j-1})$ which in turn depends only on the filtered homotopy class of f. $\square$

For a discrete group G we have a corresponding notation of a *filtered G-space* and a *filtered G-CW-complex.* The obvious G-space versions of the above lemmas hold.

3.1.4 Products of filtered complexes and spaces.

Definition 3.8. *There is a natural map:*

$$(C \otimes D) \otimes (C' \otimes D') \to (C \otimes C') \otimes (D \otimes D')$$

given by

$$c_p \otimes d_q \otimes c'_r \otimes d'_s \mapsto (-)^{qr} c_p \otimes c'_r \otimes d_q \otimes d'_s$$

Given filtered complexes F_*C and F_*D over rings R and S respectively we can form the filtered complex $F_*(C \otimes_{\mathbf{Z}} D)$ with filtration quotients given by:

$$F_r(C \otimes_{\mathbf{Z}} D) = \bigoplus_{i+j=r} F_iC \otimes_{\mathbf{Z}} F_jD$$

By analogy with the lost sign-term of algebraic surgery, for filtered complexes F_*C and F_*D we have a natural map

$$\theta^{C,D} \colon G_*(C \otimes D) \to G_*C \otimes G_*D$$

where the tensor of chain complexes in $\mathbb{D}(\mathbb{A})$ is given by the $\mathbf{Z}$-tensor products of the object chain complexes. This map is given explicitly by:

$$\theta \colon c_{p,q} \otimes d_{r,s} \mapsto (-)^{s(p+q)} c_{p,q} \otimes d_{r,s}$$

If now X and Y are filtered spaces then we define the product space $X \times Y$ by

$$(X \times Y)_r = \bigcup_{i+j=r} X_i \times Y_j$$

Observe that $F_*C(X \times Y) = F_*(C(X) \times D(X))$ so we have a map:

$$\theta^{X,Y} := \theta^{C(X),C(Y)} \colon G_*C(X \times Y) \to G_*(X) \otimes G_*(Y)$$

3.2 G-Fibrations

In this thesis we will be dealing with the more general notion of a *G-fibration*, this is a fibration with a group action on the total space:

Definition 3.9 ([Lüc86] definition 1.1). *Let G be a discrete group. A G-*fibration *consists of a G-map $E \to B$ with E a G-space and B a plain topological space (with the trivial G-action) which satisfies the G-equivariant homotopy lifting property, that is for all G-spaces X, G-map $f : X \to E$ and map $F : X \times I \to B$ such that $F(x, 0) = f(x)$ we may find a G-homotopy $H : X \times I \to E$ such that $H(x, 0) = f(x)$ and $p(H(x, t)) = F(x, t)$.*

Note that a fibration is a G-fibration with G the trivial group. For any fibration $p : E \to B$ we may form the $\pi_1(E)$-fibration $\hat{p} : \widetilde{E} \to B$ by composing p with the universal cover of E; this will be our main example of a G-fibration.

Following Lück [Lüc86] we introduce some standard terminology:

- A *map* $(\bar{f}, f) : p \to p'$ between G-fibrations $p : E \to B$ and $p' : E' \to B'$ consists of a pair of G-maps $\bar{f} : E \to E'$ and $f : B \to B'$ such that $p' \circ \bar{f} = f \circ p$.
- A G-homotopy between G-maps $f_1, f_2 : X \to Y$ between G-spaces X and Y is a homotopy $H : X \times I \to Y$ between f_1 and f_2 which is a also a G-map, where G acts on the left-hand side by acting only on X.
- A G-fibre homotopy between G-maps $f_1, f_2 \colon Z \to E$ is a G-homotopy $h : Z \times I \to E$ such that $p \circ h = p \circ f_1 = p \circ f_2$

3.2.1 Fibre transport.

This material is also from Lück [Lüc86].

Let $E \xrightarrow{p} B$ be a G-fibration and $h : f_0 \simeq f_1 : Z \to B$ a G-homotopy between two maps from a G-space Z into the base space. Then we can construct a G-fibre homotopy equivalence $f_0^*E \to f_1^*E$ as follows: The lifting problem below has a solution H

$$\begin{tikzcd} f_0^*E \arrow[r, "\hat{f}_0"] \arrow[d] & E \arrow[d, "p"] \\ f_0^*E \times I \arrow[r, "h\circ(p_{f_0}, Id)"] \arrow[ur, "H"] & B \end{tikzcd}$$

By the universal property of the pull-back we have a map of G-fibrations α_h : $f_0^*E \to f_1^*E$ such that the diagram

$$\begin{tikzcd} f_0^*E \arrow[dr, "\alpha_h"] \arrow[drr, "H(-,1)"] \arrow[ddr, "p_{f_0}"'] & & \\ & f_1^*E \arrow[r, "\hat{f}_1"'] \arrow[d, "p_{f_1}"] & E \arrow[d, "p"] \\ & Z \arrow[r, "f_1"] & B \end{tikzcd}$$

commutes. One can easily verify that the G-fibre homotopy class of α_h depends only on the G-homotopy class of h. Furthermore if now $g : Z \times I \to B$ is a G-homotopy between f_1 and a map f_2, and $h * g$ is the obvious homotopy between f_0 and f_2 then

$$\alpha_{h*g} \simeq_{p_{f_2}} \alpha_g \circ \alpha_h \tag{3.2.1}$$

If we now consider a loop x representing an element of $\pi_1(B)$ as a G-homotopy from b_0 to itself then we have a map $\alpha_x : F \to F$. This depends only on the class of x in $\pi_1(B)$ and we write $u(x) : F \to F$ for the G-homotopy class of maps given by α_x. Hence we have a map

$$u : \pi_1(B) \to [F, F]_G^{op}$$

which is a homomorphism of monoids by equation 3.2.1. In particular each $u(g)$ is a homotopy equivalence.

3.2.2 Algebraic fibre transport.

We now concentrate on the $\pi_1(E)$-fibration $\hat{p} : \widetilde{E} \to B$ for some fibration $p : E \to B$ satisfying assumption 1.2. The transport of the fibre $\hat{F} = \hat{p}^{-1}(b_0)$ along an element x is $\pi_1(E)$-equivariant, so fibre transport determines a homomorphism

$$u : \pi_1(B) \to [\hat{F}, \hat{F}]_{\pi_1(E)}^{op}$$

into the monoid of $\pi_1(E)$-homotopy classes of $\pi_1(E)$-equivariant self maps of $\hat{F}$.

Suppose now we have a $\pi_1(E)$-CW-complex $\hat{F}^{CW}$ which is $\pi_1(E)$-homotopy equivalent to $\hat{F}$. Then we can form the chain complex $C(\hat{F})$ of free $\mathbf{Z}[\pi_1(E)]$-modules. The map u determines an algebraic map:

$$u_* : \pi_1(B) \to [C(\hat{F}), C(\hat{F})]^{op}$$

which extends to a ring morphism.

$$U : \mathbf{Z}[\pi_1(B)] \to [C(\hat{F}), C(\hat{F})]^{op}$$

3.2.3 Transfer functors.

We recall the following from Lück and Ranicki [LR88].

Definition 3.10. *A representation (A, U) of a ring R in an additive category $\mathbb{A}$ is an object A in $\mathbb{A}$ together with a morphism of rings $U : R \to \mathrm{Hom}_{\mathbb{A}}(A, A)^{op}$.*

A representation (A, U) of a ring R determines a *transfer functor* $F : \mathbb{A}^{based}(R) \to \mathbb{D}(\mathbb{A})$ as follows:

$$F(R^n) = A^n$$
$$F((a_{ij}) : R^n \to R^m) = (U(a_{ij}) : A^n \to A^m)$$

We denote this functor by $- \otimes (A, U) : \mathbb{A}^{based}(R) \to \mathbb{D}(\mathbb{A})$.

Given a fibration $E \xrightarrow{p} B$, the fibre transport map

$$U : \mathbf{Z}[\pi_1(B)] \to [C(\hat{F}), C(\hat{F})]^{op}$$

constructed above gives rise to a representation $(C(\hat{F}), U)$ of the ring $\mathbf{Z}[\pi_1(B)]$ in the category $\mathbb{D}(\mathbb{A}(\mathbf{Z}[\pi_1(E)]))$. This determines the *transfer functor of the fibration*:

$$- \otimes (C(\hat{F}), U) : \mathbb{A}^{based}(\mathbf{Z}[\pi_1(B)]) \to \mathbb{D}(\mathbb{A}(\mathbf{Z}[\pi_1(E)]))$$

(in [HKR05] this functor is denoted by J^F).

If we apply such a functor to a chain complex of objects in $\mathbb{A}^{based}(\mathbf{Z}[\pi_1(B)]$ (i.e. a chain complex of based free modules) then we get a chain complex whose objects are themselves chain complexes and the differentials are given by chain complexes of chain maps.

The transfer for the product of a fibration with itself $\hat{p} \times \hat{p} : \widetilde{E} \times \widetilde{E} \to B \times B$ is given by:

$$- \otimes (C(\hat{F}) \otimes_{\mathbf{Z}} C(\hat{F}), U \otimes_{\mathbf{Z}} U) : \mathbb{A}^{based}(\mathbf{Z}[\pi_1(B) \times \pi_1(B)]) \to \mathbb{D}(\mathbf{Z}[\pi_1(E) \times \pi_1(E)])$$

By analogy with the map $\theta^{C,D}$ for filtered complexes, for all $\mathbf{Z}[\pi_1(B)]$ chain complexes C and D there is a natural map:

$$\theta^{C,D} : (C \otimes_{\mathbf{Z}} D) \otimes (C(\hat{F}) \otimes_{\mathbf{Z}} C(\hat{F}), U \otimes_{\mathbf{Z}} U) \to (C \otimes (C(\hat{F}), U)) \otimes_{\mathbf{Z}} (D \otimes (C(\hat{F}), U))$$

given by:

$$\theta^{C,D} : (c_p \otimes d_q) \otimes (f_r \otimes f'_s) \mapsto (-)^{pq} (c_p \otimes f_r) \otimes (d_q \otimes f'_s)$$

We are now in a position to state the main theorem of this chapter:

Theorem 3.11. *Let $E \to B$ be a fibration satisfying assumption 1.2, $\Delta(B) : C(\widetilde{B}) \to C(\widetilde{B}) \otimes C(\widetilde{B})$ a chain approximation of the diagonal and suppose further that we have chosen a basis for $C(\widetilde{B})$. Then there exists a filtered $\pi_1(E)$-CW-complex X, filtered $\pi_1(E)$-homotopy equivalent to $\widetilde{E}$, and a filtered chain diagonal approximation $\Delta(X)$ such that:*

1. *There exists a natural chain isomorphism $\mathcal{E} : G_*C(X) \to C(\widetilde{B}) \otimes (C(\hat{F}), U)$ on chain complexes in $\mathbb{D}(\mathbf{Z}[\pi_1(E)])$.*
2. *The diagram*

$$\begin{array}{ccc} G_*C(X) & \xrightarrow{\qquad \mathcal{E}^X \qquad} & C(\widetilde{B}) \otimes (C(\hat{F}), U) \\ \downarrow{\scriptstyle G_*C(\Delta^X)} & & \downarrow{\scriptstyle G_*C(\Delta^{\widetilde{B}}) \otimes \Delta^{\hat{F}}} \\ G_*(C(X) \otimes_{\mathbf{Z}} C(X)) & & (C(\widetilde{B}) \otimes C(\widetilde{B})) \otimes_{\mathbf{Z}} (C(\hat{F}) \otimes C(\hat{F}), U \otimes U) \\ {\scriptstyle \theta^{X,X}}\downarrow & & \downarrow{\scriptstyle \theta^{C(\widetilde{B}),C(\widetilde{B})}} \\ G_*C(X) \otimes_{\mathbf{Z}} G_*C(X) & \xrightarrow{\mathcal{E}^X \otimes \mathcal{E}^X} & (C(\widetilde{B}) \otimes (C(\hat{F}), U)) \otimes_{\mathbf{Z}} (C(\widetilde{B}) \otimes (C(\hat{F}), U)) \end{array} \tag{3.2.2}$$

commutes, where $\Delta(\hat{F})$ is the unique chain homotopy class of the diagonal map on $C(\hat{F})$.

When we say that the chain isomorphism $\mathcal{E}$ is *natural* we mean that for all such fibrations there is a specific choice of $\mathcal{E}$ which satisfies the following: Suppose that $f : B' \to B$ is a cellular map; we write $\widetilde{B'}$ for the pull-back of the universal cover of B over f. If apply the above theorem for $f^*\widetilde{E}$ then there exists some space X' filtered $\pi_1(E)$-homotopy equivalent to $f^*\widetilde{E}$ and a natural map $\mathcal{E}' : G_*C(X') \to C(\widetilde{B'}) \otimes (C(\hat{F}), U)$. Then naturality of the maps $\mathcal{E}$ and $\mathcal{E}'$ means that the diagram:

$$\begin{array}{ccc} C(X') & \xrightarrow{\qquad\qquad} & C(X) \\ {\scriptstyle \mathcal{E}'}\downarrow & & \downarrow{\scriptstyle \mathcal{E}} \\ C(\widetilde{B'}) \otimes (C(\hat{F}), U) & \xrightarrow{C(\tilde{f}) \otimes (C(\hat{F}), U)} & C(\widetilde{B}) \otimes (C(\hat{F}), U) \end{array} \tag{3.2.3}$$

commutes in $\mathbb{D}(\mathbf{Z}[\pi_1(E)])$.

The rest of this chapter is devoted to the proof of this theorem; none of the material is used elsewhere in the thesis.

3.2.4 Trivializations of the bundle.

The first thing that we need to do is to investigate the pull-back of a bundle over an attaching map on the base space. We recall again from [Lüc86]:

Definition 3.12. *Let (D, d_0) be a pointed contractible space, $f : D \to B$ a map and ξ a path from $f(d_0)$ to b_0. Given any homotopy $h : D \times I \to B$ between the constant map taking D to b_0 and f such that $h(d_0, i) = \xi(i)$ we may construct a map*

$$T_h(f, \xi) : D \times \hat{F} \to E$$

by $\alpha_h : D \times \hat{F} \to f^*E$ composed with the map $f^*E \to E$. Up to G-fibre homotopy this map is independent of the choice of h and we write $T(f, \xi)$ for the G-fibre homotopy class of $T_h(f, \xi)$. Observe that $p \circ T(f, \xi) = f \circ pr_D : D \to B$, so for a map of pairs $f : (D^k, S^{k-1}) \to (B_k, B_{k-1})$ we have a corresponding map

$$T(f, \xi) : (D^k, S^{k-1}) \times \hat{F} \to (\widetilde{E}_k, \widetilde{E}_{k-1})$$

One may think of the map $T(f, \xi)$ as "trivialization of the bundle along the path ξ". Lück proves the following:

Lemma 3.13. *Let $h : (D^k, S^{k-1}) \times I \to (B_k, B_{k-1})$ be a homotopy between two maps $f_1, f_2 : D^k \to B$ and ξ_1, ξ_2 paths from $f_1(d_0)$ respectively $f_2(d_0)$ to b_0. Write ξ for the path $h(d_0, -)$. Then the following diagram commutes up to homotopy of pairs:*

$$\begin{tikzcd}
(D^k, S^{k-1}) \times \hat{F} \arrow[r, "T(f_1,\xi_1)"] \arrow[d, "u(\xi_1 * \xi^- * \xi_2^-) \times Id"'] & (E_k, E_{k-1}) \\
(D^k, S^{k-1}) \times \hat{F} \arrow[ur, "T(f_2,\xi_2)"'] &
\end{tikzcd}$$

3.3 Describing the CW-complex of a fibration.

From this point on the proof becomes rather more algebraic and involves chain maps. We consider all chain maps to be defined only up to chain homotopy. When we write $C(f)$ for maps $f \colon X \to Y$ of G-CW-complexes what we mean is the chain homotopy class of some G-CW-approximation to f. This is well defined since all such approximations yield chain homotopic chain maps. In the case of filtered maps we will similarly take filtered G-CW-approximations. We will identify $C(D^k, S^k)$ with the chain complex which is $\mathbf{Z}$ concentrated in dimension k; furthermore we will write $C((D^k, S^{k-1}) \times \hat{F})$ as $S^k C(\hat{F})$ using this identification. We also consider (D^k, S^{k-1}) to be a filtered space with $(k-1)$-th filtration S^k and k-th filtration D^k. In this case we identify $G_k C((D^k, S^{k-1}) \times \hat{F}^{CW})$ with $C(\hat{F}^{CW})$.

We will now fix some fibration $p : E \to B$ satisfying assumption 1.2. The k-cells in B will be indexed by a set J_k and we will denote the attaching maps by

$(Q(j), q(j)) : (D^k, S^{k-1}) \to (B_k, B_{k-1})$, $j \in J_k$; we also assume that the set J_k is ordered.

Suppose we have a map $(R, r) : (D^k, S^{k-1}) \to (B_k, B_{k-1})$ and a path ξ from $R(d_0)$ to b_0. Then we can find a unique map $(\widetilde{R}, \widetilde{r}, \xi) : (D^k, S^{k-1}) \to (\widetilde{B}_k, \widetilde{B}_{k-1})$ such that

1. The composition of $(\widetilde{R}, \widetilde{r})$ with the universal cover projection of B is (R, r).
2. There exists a lift $\widetilde{\xi}$ of the path ξ to the universal cover which goes from $\widetilde{R}(d_0)$ to $\widetilde{b}_0 \in \widetilde{B}$ the base-point.

We have a chain map

$$C(\widetilde{R}, \widetilde{r}, \xi) : S^k\mathbf{Z} \to C(\widetilde{B}_k, \widetilde{B}_{k-1})$$

Taking the tensor product with $\mathbf{Z}[\pi_1(B)]$ (with $\pi_1(B)$ acting trivially on $\mathbf{Z}$) and ignoring all but the k'th term we have a chain map

$$\bar{C}(R, r, \xi) := C(\widetilde{R}, \widetilde{r}, \xi)_k \otimes \mathbf{Z}[\pi_1(B)] : \mathbf{Z}[\pi_1(B)] \to C(\widetilde{B})_k$$

We can use this construction to define a basis for $C(\widetilde{B})_k$. Choosing paths η_j for $j \in J_k$ from $Q(j)(d_0)$ to the base-point b_0 we have an isomorphism:

$$\bigoplus_{j \in J_k} \mathbf{Z}[\pi_1 B] \xrightarrow{\bigoplus \bar{C}(Q(j), q(j), \eta_j)} C(\widetilde{B})_k$$

which is effectively a basis for $C(\widetilde{B})_k$ since we've assumed that J_k is ordered. From now on we assume that we've chosen paths η_j which define such a basis for each $C(\widetilde{B})_k$.

If $k \geq 2$ then by the relative Hurewicz theorem we have an isomorphism:

$$\pi_k(\widetilde{B_k}, \widetilde{B}_{k-1}, b_0) \cong C(\widetilde{B}_k, \widetilde{B}_{k-1})$$

Furthermore we can apply the long exact sequence of a fibration to the covering map $\widetilde{B} \to B$ to get an isomorphism of groups:

$$H : \pi_k(B_k, B_{k-1}, b_0) \xrightarrow{\cong} C(\widetilde{B}_k, \widetilde{B}_{k-1}) \tag{3.3.1}$$

We can see this map explicitly (as well as the left $\pi_1(B)$-action on the left-hand side) by using the above construction: Choose (R, r) and ξ as above. We can always approximate ξ by a path lying in B_{k-1} so we assume that ξ is such a path. Regarding $\xi^- \circ r|_{\{d_0\}} : \{d_0\} \times I \to B$ as a homotopy we can use the homotopy extension property of the pairs (S^{k-1}, d_0) and (D^k, d_0) to construct a homotopy of pairs

$$H : (D^k, S^{k-1}) \times I \to (B_k, B_{k-1})$$

such that the diagram

$$\begin{tikzcd}
(D^k, S^{k-1}) \arrow[r] \arrow[d, "{(R,r)}"'] & (D^k, S^{k-1}) \times I \arrow[dl, "H"'] \\
B & d_0 \times I \arrow[l, "\xi^{-} \circ r|_{d_0}"'] \arrow[u, "i \times Id"']
\end{tikzcd}$$

commutes. Then $H(-,1) : (D^k, S^{k-1}) \to (B_k, B_{k-1})$ is an element of $\pi_k(B_k, B_{k-1}, b_0)$. We denote this element by (R, r, ξ). Note that this depends only on the homotopy class of ξ modulo the end-points rather than on ξ itself, in particular it doesn't depend on any approximation we may have taken for ξ. The image of (R, r, ξ) under the isomorphism 3.3.1 is given by the image of the generator $1 \in \mathbf{Z}[\pi_1 B]$ under the map $\bar{C}(R, r, \xi)$. The action of an element x of $\pi_1(B)$ on an element (R, r) of $\pi_1(\widetilde{B}_k, \widetilde{B}_{k-1}, b_0)$ is given by

$$x.(R, r) = (R, r, x^{-1})$$

(note that the inverse is required to make this into a *left* action)

Definition 3.14. *A* CW*-model* $(X, \psi^X, \hat{F}^{CW}, \lambda^X, \mathcal{E}^X)$ *for the total space* $\widetilde{E}$ *of a fibration* $p\colon E \to B$ *satisfying assumption 1.2 consists of*

- *A* $\pi_1(E)$*-*CW*-complex* $\hat{F}^{CW}$ *and a* $\pi_1(E)$*-homotopy equivalence* $\lambda^X : \hat{F}^{CW} \to \hat{F}$.
- *A filtered* $\pi_1(E)$*-*CW*-complex* X *and a* $\pi_1(E)$*-homotopy equivalence* $\psi^X : \widetilde{E} \to X$.
- *For each* k *an isomorphism of chain complexes:*

$$\mathcal{E}^X_k : G_k C(\widetilde{X}) \xrightarrow{\cong} C(\widetilde{B})_k \otimes (C(\hat{F}^{CW}), U)$$

where $- \otimes (C(\hat{F}^{CW}), U) : \mathbb{A}^{based}(\mathbf{Z}[\pi_1(B)]) \to \mathbb{D}(\mathbf{Z}[\pi_1(B))$ *is the transfer functor defined in section 3.2.3 (recall that we've chosen a basis for* $C(\widetilde{B})$ *so the right-hand side is well-defined)*

To clear up the notation we define

$$T^{CW}(f, \xi) := \psi^X \circ T(f, \xi) \circ (Id \times \lambda^X) : D \times \hat{F}^{CW} \to X$$

for maps $f : D \to B$ *from a contractible space* D *and paths* $\xi : f(d_0) \to b_0$.

In addition the following condition must be satisfied: For each attaching map $(Q(j), q(j)) : (D^k, S^{k-1}) \to (B_r, B_{r-1})$ *and basis path* η_j *the trivialization*

$$T^{CW}(Q(j), \eta_j) : (D^k, S^k) \times \hat{F} \to (X_k, X_{k-1})$$

satisfies

$$\mathcal{E}_k^X \circ G_* C(T^{CW}(Q(j),\eta_j))_k = \bar{C}(Q(j),q(j),\eta_j) \otimes (C(\hat{F}),U)) \\ : C(\hat{F}) \to C(\widetilde{B})_k \otimes (C(\hat{F}),U). \tag{3.3.2}$$

Note that we do not insist that the map $\mathcal{E}_* : C(X) \to C(\widetilde{B}) \otimes (C(\hat{F}),U)$ is a chain map (of chain complexes in $\mathbb{D}(\mathbf{Z}[\pi_1(E)]))$, although it will turn out that this is the case (Theorem 3.17). Observe also that the condition 3.3.2 is equivalent to saying that:

$$\bigoplus_{j \in J_k} G_* C(T^{CW}(Q(j),\eta_j)) = \mathcal{E}^{-1} : C(\widetilde{B})_k \otimes (C(\hat{F}),U) \to G_k C(X) \tag{3.3.3}$$

since $\bigoplus_{j\in J_k} \bar{C}(Q(j),q(j),\eta_j)$ is by definition the based identity map.

Having made such a definition we ought to show that such things exist:

Theorem 3.15. *Given a fibration $p : E \to B$ satisfying assumption 1.2 then there exists a CW-model $(X,\psi^X,\hat{F}^{CW},\lambda^X,\mathcal{E}^X)$ for $\widetilde{E}$.*

Proof. This is essentially a "filtered" version of the argument of [Lüc86] section 7A. We first choose any $\pi_1(E)$-homotopy equivalence $\lambda^X : \hat{F}^{CW} \to \hat{F}$ from some $\pi_1(E)$-space $\hat{F}^{CW}$. We now proceed by induction on the filtration of E. Since $\widetilde{E}_0$ is just a disjoint union of spaces homotopy equivalent to $\hat{F}$ we define X_0 to be $\#(J_0)$ copies of $\hat{F}^{CW}$ and $\psi_0^X : \widetilde{E}_0 \to X_0$ the obvious $\pi_1(E)$-homotopy equivalence. We can now simply define the map $\mathcal{E}_0$ to be the inverse of $\bigoplus_{j\in J_0} C(T^{CW}(Q(j),\eta_j))$ so it clearly satisfies the condition.

We proceed to the inductive step: Suppose that we have already constructed a CW-model $(X_{k-1},\psi_{k-1}^X,\hat{F}^{CW},\lambda^X,\mathcal{E}^X)$ for E_{k-1}. We have a $\pi_1(E)$-push-out diagram given by the attaching maps $(Q(j),q(j))$:

$$\begin{tikzcd} Q(j)^*\widetilde{E} \arrow[r] & \widetilde{E}_k \\ q(j)^*\widetilde{E} \arrow[r] \arrow[u] & \widetilde{E}_{k-1} \arrow[u] \end{tikzcd}$$

which we extend using the trivializations $T^{CW}(Q(j),\eta_j)$ to a diagram:

$$\begin{tikzcd} \bigsqcup_{J_k} D^k \times \hat{F}^{CW} \arrow[r, "\sqcup T(Q(j){,}\eta_j)"] & Q(j)^*\widetilde{E} \arrow[r] & E_k & \\ \bigsqcup_{J_k} S^k \times \hat{F}^{CW} \arrow[r] \arrow[u] & q(j)^*\widetilde{E} \arrow[r] \arrow[u] & E_{k-1} \arrow[r, "\psi_{k-1}^X"] \arrow[u] & X_{k-1} \end{tikzcd}$$

We choose a $\pi_1(E)$-CW-approximation ϕ_k to the bottom row and write h_k for the homotopy. We define X_k to be the $\pi_1(E)$-push-out

$$\begin{tikzcd}
\bigsqcup_{J_k} D^k \times \hat{F}^{CW} \arrow[r, "\sqcup T(Q(j),\eta_j)"] & X_k \\
\bigsqcup_{J_k} S^k \times \hat{F}^{CW} \arrow[r, "\phi_k"] \arrow[u] & X_{k-1} \arrow[u]
\end{tikzcd}$$

Observe that X_k has the structure of a $\pi_1(E)$-CW-complex with attaching maps given by those for X_{k-1} and the maps $D^k \times F^{CW} \to X_k$. Observe that $X_{k-1} \subset X_k$ so X_k inherits the filtration of X_{k-1}. We are now required to find a filtered $\pi_1(E)$-homotopy equivalence $\widetilde{E} \to X$. We have a diagram of maps (c.f. [Lüc86] page 115):

$$\begin{tikzcd}
\bigsqcup_{j\in J_k} Q(j)^*\widetilde{E} \arrow[d] & \bigsqcup_{j\in J_k} q(j)^*\widetilde{E} \arrow[l] \arrow[r] \arrow[d] & \widetilde{E}_{k-1} \arrow[d, "\psi^X_{k-1}"'] \\
\bigsqcup_{j\in J_k} D^k \times \hat{F}^{CW} \arrow[d, "i_0"'] & \bigsqcup_{j\in J_k} S^{k-1} \times \hat{F}^{CW} \arrow[l] \arrow[r, "\sqcup T^{CW}(Q(j),\eta_j)"] \arrow[d, "i_0"'] & X_{k-1} \arrow[d, "Id"'] \\
\bigsqcup_{j\in J_k} D^k \times \hat{F}^{CW} \times I & \bigsqcup_{j\in J_k} S^{k-1} \times \hat{F}^{CW} \times I \arrow[l] \arrow[r, "h_k"] & X_{k-1} \\
\bigsqcup_{j\in J_k} D^k \times \hat{F}^{CW} \arrow[u, "i_1"] & \bigsqcup_{j\in J_k} S^{k-1} \times \hat{F}^{CW} \arrow[l] \arrow[r, "\phi_k"] \arrow[u, "i_1"] & X_{k-1} \arrow[u, "Id"']
\end{tikzcd}$$

The total space $\widetilde{E}_k$ is the push-out of the top row, the space X_k is the push-out of the bottom row so we are required to show that each of the triplets of vertical maps induces a homotopy equivalence of the filtered spaces given by the push-outs of the rows (the filtrations on the push-outs of the middle two row are given in the obvious way). The fact that all of the left-hand horizontal maps have the homotopy extension property implies that the induced maps are homotopy equivalences (see Brown [Bro68] page 249)). We can now apply the equivariant version of lemma 3.6 to see that the induced maps are $\pi_1(E)$-homotopy equivalences of filtered spaces. Hence we have constructed the map ψ_j^X.

We have an isomorphism:

$$S^{-k}C(\bigsqcup_{J_k}(D^k, S^{k-1}) \times \hat{F}^{CW}) \cong \bigoplus_{J_k} C(\hat{F}) \xrightarrow{\oplus T(Q(j),\eta_j)} S^{-k}C(X_k, X_{k-1}) \cong G_k C(X)$$

Again we simply define $\mathcal{E}_k^X$ to be the inverse of this map so as to satisfy the alternative formulation of the required condition (equation 3.3.3). □

The aim now is to prove the following proposition which relates algebra and topology for a CW-model. In effect it generalizes the condition 3.3.2 to all maps $(R,r):(D^k,S^{k-1})\to(B_k,B_{k-1})$ rather than just those that represent the basis. It allows us to put the most technical calculations involving the fibre transport in one place.

Proposition 3.16. *Given a map $(R,r):(D^k,S^{k-1})\to(B_k,B_{k-1})$ and a path ξ from $f(d_0)$ to b_0 then*

$$\mathcal{E}_k^X\circ G_*C(T^{CW}(R,\xi))_k=\bar{C}(R,r,\xi)\otimes(C(\hat{F}),U)\colon C(\hat{F}^{CW})\to C(\widetilde{B})\otimes(C(\hat{F}),U)$$

Proof. This is essentially proved (at least in the case $k\geq 2$) in [Lüc86] section 7D, using slightly different language. The author prefers a slightly different, more algebraic proof which is that presented here: We first demonstrate that the proposition holds for $(R,r)=(Q(j),q(j))$ for some $j\in J_k$. We will write the path ξ as η_j*x for some loop x, which we can do because everything depends only on the homotopy class of ξ modulo $\{0,1\}$. By lemma 3.13 we have a commutative diagram:

$$\begin{tikzcd}
(D^k,S^{k-1})\times\hat{F}^{CW} \arrow[r, "T(Q(j)\text{,}\eta_j)"] \arrow[d, "u(x^{-1})"'] & (\widetilde{E}_k,\widetilde{E}_{k-1}) \\
(D^k,S^{k-1})\times\hat{F}^{CW} \arrow[ur, "T(Q(j)\text{,}x*\eta_j)"'] &
\end{tikzcd}$$

Composing with ψ^X and λ^X and taking associated chain complexes we get:

$$\begin{tikzcd}
C(\hat{F}^{CW}) \arrow[r, "G_*C(T^{CW}(Q(j)\text{,}\eta_j))"] \arrow[d, "(x^{-1})\otimes(C(\hat{F})\text{,}U)"'] & G_kC(X) \\
C(\hat{F}^{CW}) \arrow[ur, "G_*C(T^{CW}(Q(j)\text{,}\eta_j*x))"'] &
\end{tikzcd}$$

By the definition of a CW-model the top map is given by $\bar{C}(Q(j),q(j),\eta_j)\otimes(C(\hat{F}^{CW}),U)$ so we get

$$\begin{aligned}
G_*C(T^{CW}(Q(j),\eta_j*x)) &= G_*C(T^{CW}(Q(j),\eta_j))\,\fatsemi\,((x)\otimes(C(\hat{F}^{CW}),U))\\
&= (x^{-1}.\bar{C}(Q(j),q(j),\eta_j))\otimes(C(\hat{F}^{CW}),U)\\
&= \bar{C}(Q(j),q(j),\eta_j*x)\otimes(C(\hat{F}^{CW}),U)
\end{aligned}$$

This completes the proof of the $(R,r)=(Q(j),q(j))$ case, it also covers the $k=0$ case since all maps take this form. We now tackle the general case; we will deal with the case $k=1$ at the end so without further ado we assume that $k\geq 2$. The first observation which we make is that the trivialization of (R,r)

along the path ξ is $\pi_1(E)$-homotopic to the trivialization of the map (R,r,ξ) : $(D^k, S^{k-1}) \to (B_k, B_{k-1}, b_0)$ along the trivial path from b_0 to itself. By lemma 3.13 the map $T^{CW}(R,r)$ depends (up to homotopy of pairs) only on the class $(R,r,\xi) \in \pi_k(B_k, B_{k-1}, b_0)$. Therefore it is sufficient to prove that the lemma holds for maps $(R,r) : (D^k, S^{k-1}) \to (B_k, B_{k-1}, b_0)$ representing elements of $\pi_1(B_k, B_{k-1}, b_0)$. We have a topologically defined map:

$$\Omega : \pi_1(B_k, B_{k-1}, b_0) \to [C(\hat{F}), C(\widetilde{B})_k \otimes (C(\hat{F}), U)]$$

$$(R,r) \mapsto (\mathcal{E}_k^X \circ G_* C(T^{CW}(R, Id))$$

In order to prove the lemma we must show that this coincides with the map given by

$$(R,r) \mapsto (C(R,r) \otimes (\hat{F}^{CW}, U))$$

We do this in two steps:

1. We show that Ω is a homomorphism.

2. We show that the required formula holds for generators of $\pi_1(B_k, B_{k-1}, b_0)$.

For the first step, let (R,r) and (R',r') be two maps from (D^k, S^{k-1}) to (B_k, B_{k-1}, b_0) and let $\nabla : (D^k, S^{k-1}) \vee (D^{k-1}, S^{k-1}) \to (D^k, S^{k-1})$ be the map inducing the group structure in $\pi_1(B_k, B_{k-1}, b_0)$. Then we have a commutative diagram:

$$\begin{tikzcd}
(D^k, S^{k-1}) \times \hat{F}^{CW} \arrow[dr, "T^{CW}((R\vee R')\circ\nabla, b_0)"] & \\
((D^k, S^{k-1}) \vee (D^k, S^{k-1})) \times \hat{F}^{CW} \arrow[u, "\nabla\times Id"'] \arrow[r, "T(R,b_0)\vee T(R',b_0)"] & (X_k, X_{k-1}) \\
((D^k, S^{k-1}) \sqcup (D^k, S^{k-1})) \times \hat{F}^{CW} \arrow[u] \arrow[ur, "T^{CW}(R,b_0)\sqcup T^{CW}(R',b_0)"'] &
\end{tikzcd}$$

Taking associated chain complexes we see that $\Omega((R,r) + (R',r')) = \Omega(R,r) + \Omega(R',r')$ as required for the first step. For the second step, observe that $\pi_1(B_k, B_{k-1}, b_0)$ has a basis given by element of the form $\xi.(Q(j), q(j), \eta_j)$ for $j \in J_k$ and $\xi \in \pi_1(B)$ (equation 3.3.1). However proving that the required formula holds for these maps is equivalent to proving that it holds for maps $(Q(j), q(j))$ and paths $\eta_j.\xi$, which we've already covered at the beginning of the proof. This completes the proof of the $k \geq 2$ case.

We now turn our attention to the case $k = 1$. A map $(D^1, S^0) \to (B_1, B_0)$ represents a path between vertices in B_1. Any such path can be represented by the composition of elementary paths given by the attaching maps $(Q(j), q(j))$ for

$j \in J_1$. We will write the composition of paths (R, r) and (R', r') as $(R * R', r * r')$. By considering the lifts of the relevant paths we deduce that

$$\bar{C}(R * R', r * r', \xi) = \bar{C}(R, r, \xi) + \bar{C}(R', r', R * \xi)$$

for paths (R, r) and (R', r') such that $R(1) = R'(0)$ and a path ξ in B from $R(0)$ to b_0. In order to complete the proof we are required to show two things:

1. That $C(T^{CW}(R * R', \xi)) = C(T^{CW}(R, \xi)) + C(T^{CW}(R', R * \xi)) : \mathbf{Z}[\pi_1(B)] \to C(\widetilde{B})_k \otimes (C(\hat{F}), U)$ for $(R, r), (R', r'), \xi$ as above.

2. That the lemma holds for the maps $(Q(j), q(j))$ for all $j \in J_1$ and paths $\eta_j.\xi$ for $\xi \in \pi_1(B)$.

We have already shown that the second statement is true at the beginning of the proof. For the first statement we have a commutative diagram:

$$\begin{tikzcd}
(D^1, S^0) \times \hat{F}^{CW} \arrow[drr, "T^{CW}(R * R', \xi)"] & & \\
((D^1, S^0) * (D^1, S^0)) \times \hat{F}^{CW} \arrow[u, "* \times ID"'] \arrow[rr, "T(R,\xi) * T(R', R*\xi)"] & & (X_1, X_0) \\
((D^1, S^0) \sqcup (D^1, S^0)) \times \hat{F}^{CW} \arrow[u] \arrow[urr, "T^{CW}(R,\xi) \sqcup T^{CW}(R', R*\xi)"'] & &
\end{tikzcd}$$

Taking chain complexes yields the required formula and hence we've proved the lemma in the case $k = 1$. This completes the proof of the lemma. $\square$

We now prove, as promised, that the map $\mathcal{E}$ is a chain map.

Theorem 3.17. *Let X be a CW-model for the total space of a fibration $p : E \to B$. Then the maps $\mathcal{E}_k$ induce an isomorphism of chain complexes in $\mathbb{D}(\mathbf{Z}[\pi_1 B])$:*

$$\mathcal{E} : G_* C(X) \to C(\widetilde{B}) \otimes (C(\hat{F}), U)$$

Proof. We are required to show that the differential in $G_* C(X)$ is given by

$$G_k d = \mathcal{E}_{k-1}^{-1} \circ (d_k^B \otimes (C(\hat{F}), U) \circ \mathcal{E}_k : G_k C(X) \to G_{k-1} C(X)$$

under the identification $\mathcal{E}$ of $G_k C(X)$ with $C(\widetilde{B})_k \otimes (C(\hat{F}), U)$. We write $(\bar{q}(j), .)$ for the composition

$$(D^{k-1}, S^{k-2}) \to (S^{k-1}, s_0) \xrightarrow{q(j)} (B_{k-1}, B_{k-2})$$

The basis of $C(\widetilde{B})$ identifies it with $\bigoplus_{J_k} \mathbf{Z}[\pi_1(B)]$, moreover we can describe the map $d_k : C(\widetilde{B})_k \to C(\widetilde{B})_{k-1}$ with respect to this identification as:

$$d_k = \bigoplus_{J_k} \bar{C}(\bar{q}(j), ., \eta_j) : C(\widetilde{B})_k \to C(\widetilde{B})_{k-1}$$

For each $j \in J_k$ we have a commutative diagram:

$$\begin{tikzcd}
C((D^k, S^{k-1}) \times \hat{F}) \arrow[rr, "C(T^{CW}(Q(j),\eta_j))"] \arrow[d, "d\otimes 1"'] & & C(X_k, X_{k-1}) \arrow[dd, "d"] \\
SC((S^{k-1}, s_0) \times F^{CW}) \arrow[d, "\cong"'] & & \\
SC((D^{k-1}, S^{k-2}) \times \hat{F}^{CW}) \arrow[rr, "C(T^{CW}(\bar{q}(j),\eta_j))"] & & SC(X_{k-1}, X_{k-2})
\end{tikzcd}$$

Both the top-left and bottom-left groups can be identified with $C(\hat{F}^{CW})$ in a canonical way, in this case the composition of the maps on the left-hand side is given by $(-)^{r+k} : C(\widehat{F}^{CW}) \to C(\widehat{F}^{CW})$ (the sign term comes from that in the definition of the tensor product of two chain complexes).

Looking at the graded complex we get the diagram:

$$\begin{tikzcd}
C(\hat{F}^{CW}) \arrow[r, "G_kC(T^{CW}(Q(j),\eta_j))"] \arrow[d, "Id"'] & G_kC(X) \arrow[d, "G_kd"] \\
C(\hat{F}^{CW}) \arrow[r, "G_kC(T^{CW}(\bar{q}(j),\eta_j))"] & G_{k-1}C(X)
\end{tikzcd}$$

Summing over all $j \in J_k$ and extending to the right we get a diagram

$$\begin{tikzcd}
\bigoplus_{j\in J_k} C(\hat{F}) \arrow[r, "\oplus G_kC(T^{CW}(Q(j),\eta_j))"] \arrow[d, "Id"'] & G_kC(X) \arrow[r, "\mathcal{E}_k"] \arrow[d, "G_kd"'] & C(\widetilde{B})_k \otimes C(\hat{F}) \arrow[d, "d_k\otimes(C(\hat{F}),U)"] \\
\bigoplus_{j\in J_k} C(\hat{F}) \arrow[r, "\oplus G_kC(T^{CW}(\bar{q}(j),\eta_j))"] & G_kC(X) \arrow[r, "\mathcal{E}_{k-1}"] & C(\widetilde{B})_{k-1} \otimes C(\hat{F})
\end{tikzcd}$$

The left-hand square clearly commutes from the above. The composition along the the top is given by

$$\begin{aligned}
\mathcal{E}_k \circ \bigoplus_{j\in J_k} G_kC(T^{CW}(Q(j), \eta_j)) &= \bigoplus_{j\in J_k} \bar{C}(Q(j), q(j), \eta_j) \otimes (C(\hat{F}), U) \\
&= Id \\
&: \bigoplus_{j\in J_k} C(\hat{F}) = C(\widetilde{B})_k \otimes (C(\hat{F}), U) \to C(\widetilde{B})_k \otimes (C(\hat{F}), U)
\end{aligned}$$

Applying proposition 3.16 to the maps $\bar{q}(j)$ we see that the composition along the bottom is given by:

$$\begin{aligned}
\mathcal{E}_{k-1} \circ \bigoplus_{j\in J_k} G_kC(T^{CW}(\bar{q}(j), \eta_j)) &= \bigoplus_{j\in J_k} \bar{C}(\bar{q}(j), ., \eta_j) \otimes (C(\hat{F}), U) \\
&= d_k \otimes (C(\hat{F}^{CW}), U)
\end{aligned}$$

$$: \bigoplus_{j\in J_k} C(\hat{F}) = C(\widetilde{B})_k \otimes (C(\hat{F}),U) \to C(\widetilde{B})_k \otimes (C(\hat{F}),U)$$

Hence the outer square commutes so we deduce that the right-hand square commutes. Therefore $\mathcal{E}$ is a chain map as required. $\square$

3.3.1 Finishing the proof.

Proof of theorem 3.11. By theorem 3.15 there exists a CW-model $(X, \psi^X, \hat{F}^{CW}, \lambda^X, \mathcal{E}^X)$ for E and we take X to be the filtered space here. By theorem 3.17 the chain maps $\mathcal{E}_k$ induce the required isomorphism of $G_*C(\widetilde{X})$ with $C(\widetilde{B}) \otimes (C(\hat{F}),U)$.

We must establish that this construction satisfies the required naturality property 3.2.3. Suppose $f : B' \to B$ is a cellular map and let $\widetilde{B'}$ be the pullback of the universal cover of B over f as before. There exists a CW-model $(X', \psi^{X'}, \hat{F}^{CW}, \lambda^X, \mathcal{E}^{X'})$. Let $(Q'(j), q(j)) : (D^k, S^{k-1}) \to (B'_k, B'_{k-1})$ be the attaching maps for B' and let η'_j be some basis paths. For each k we have a diagram of chain complexes:

$$\begin{tikzcd}
\bigoplus_{J'_k} C(\hat{F}^{CW}) \arrow[r, "\oplus G_k(T^{CW}(Q'(j),\eta_j))"] \arrow[d, "="] & G_kC(X') \arrow[r] \arrow[d, "\mathcal{E}'"] & G_kC(X) \arrow[d, "\mathcal{E}"] \\
\bigoplus_{J'_k} C(\hat{F}^{CW}) \arrow[r, "\tilde{C}(Q(j),q(j),\eta_j)\otimes(C(\hat{F}),U)"] & C(\widetilde{B'})_k \otimes (C(\hat{F}),U) \arrow[r, "C(\tilde{f})\otimes(C(\hat{F}),U)"] & C(\widetilde{B})_k \otimes (C(\hat{F}),U)
\end{tikzcd}$$

The left-hand square commutes from the definition of a CW-model and the outer square commutes by Proposition 3.16. However the two left-hand horizontal maps are isomorphism so the right-hand square must commute. Hence the CW-model construction satisfies the required naturality condition.

The only thing left for us to do it to find the filtered approximation to the diagonal and to show that is has the required properties. Let $h : B \times I \to B \times B$ be a homotopy between $diag^B$ and a cellular map Δ^B. Consider the following diagram:

$$\begin{tikzcd}
\widetilde{E} \arrow[rr, "diag^{\widetilde{E}}"] \arrow[d] & & \widetilde{E} \times \widetilde{E} \arrow[d, "\hat{p}\times\hat{p}"] \\
\widetilde{E} \times I \arrow[r, "\hat{p}\times Id"] & B \times I \arrow[r, "h"] & B \times B
\end{tikzcd}$$

The map $p \times p : \widetilde{E} \times \widetilde{E} \to B \times B$ is a $\pi_1(E)$-fibration with $\pi_1(E)$ acting via the diagonal action and the above diagram is a homotopy lifting problem so there exists a map $H : \widetilde{E} \times I \to \widetilde{E} \times \widetilde{E}$ such that

$$\begin{tikzcd}
\widetilde{E} \arrow[rr, "diag^{\widetilde{E}}"] \arrow[d] & & \widetilde{E} \times \widetilde{E} \arrow[d, "\hat{p}\times\hat{p}"] \\
\widetilde{E} \times I \arrow[r, "\hat{p}\times Id"] \arrow[urr, "H"] & B \times I \arrow[r, "h"] & B \times B
\end{tikzcd}$$

commutes. We define the filtered map $\Delta^{\widetilde{E}} := H(-,1) : \widetilde{E} \to \widetilde{E} \times \widetilde{E}$ and define $\Delta^X : X \to X \times X$ to be the composition of filtered maps:

$$X \xrightarrow{(\psi^X)^{-1}} \widetilde{E} \xrightarrow{\Delta^{\widetilde{E}}} \widetilde{E} \times \widetilde{E} \xrightarrow{\psi^X \times \psi^X} X \times X$$

(defined up to homotopy) We have a CW-model for the $\pi_1(E) \times \pi_1(E)$-fibration $\widetilde{E} \times \widetilde{E}$ given by

$$(X \times X, \psi^X \times \psi^X, \hat{F}^{CW} \times \hat{F}^{CW}, \lambda^X \times \lambda^X, (\theta^{C(\widetilde{B}),C(\widetilde{B})})^{-1}(\mathcal{E} \otimes_{\mathbf{Z}} \mathcal{E}) \circ \theta^{X,X}) \quad (3.3.4)$$

We are required to show that diagram 3.2.2 commutes. Fix $k > 0$. For any $j \in J_k$ we choose a homotopy h between the map $Q(j) : D^k \to B_k$ and the map taking all of D^k to the base point d_0 and such that $h(i, d_0) : I \to B_k$ represents the path η_j. Choosing a lift H of the map $h \circ (i_{D^k \times I}) : D^k \times \hat{F} \times I \to B_k$ such that

$$\begin{tikzcd}
D^k \times \hat{F} \arrow[r, "pr_{\hat{F}}"] \arrow[d] & \widetilde{E} \arrow[d, "\hat{p}"] \\
D^k \times \hat{F} \times I \arrow[r, "h\circ(i_{D^k \times I})"] \arrow[ur, "H"] & B
\end{tikzcd}$$

commutes allows us to construct an explicit trivialization:

$$T^h(Q(j), \eta_j) : (D^k, S^k) \times \hat{F} \to (\widetilde{E}_k, \widetilde{E}_{k-1})$$

(the map H is required to explicitly construct a choice of α_h). Similarly the map $H \circ \Delta^{\widetilde{E}}$ is a lift of the homotopy composing h with the diagonal approximation $h \circ \Delta^B$ so we have an explicit trivialization:

$$T^{\Delta^b \circ h}(\Delta^B \circ Q(j), \eta_j \times \eta_j) : (D^k, S^k) \times (\hat{F} \times \hat{F}) \to ((\widetilde{E} \times \widetilde{E})_k, (\widetilde{E} \times \widetilde{E})_{k-1})$$

These explicit maps allow us to see that the following diagram commutes:

$$\begin{tikzcd}
(D^k, S^{k-1}) \times \hat{F}^{CW} \arrow[r, "Id \times diag^F"] \arrow[d, "T^h(Q(j),\eta_j)"'] & (D^k, S^{k-1}) \times (\hat{F}^{CW} \times \hat{F}^{CW}) \arrow[d, "T^{\Delta^B \circ h}(\Delta^B \circ Q(j), \eta_j \times \eta_j)"] \\
\widetilde{E} \arrow[r, "\Delta^E"] \arrow[d] & \widetilde{E} \times \widetilde{E} \arrow[d] \\
X \arrow[r, "\Delta^X"] & X \times X
\end{tikzcd}$$

We can now take associated chain complexes of this diagram to yield:

$$\begin{tikzcd}
C(\hat{F}) \arrow[r, "C(\Delta^{\hat{F}})"] \arrow[d, "G_k C(T^{CW}(Q(j),\eta_j))"'] & C(\hat{F} \times \hat{F}) \arrow[d, "G_k C(T^{CW}(\Delta^B \circ Q(j),\eta_j \times \eta_j))"] \\
G_k C(X) \arrow[r, "G_k C(\Delta^X)"] & G_k C(X \times X)
\end{tikzcd}$$

and sum over all $j \in J_k$ to obtain:

$$\begin{tikzcd}
C(\widetilde{B})_k \otimes (C(\hat{F}), U) \arrow[r, "Id \otimes C(\Delta^{\hat{F}})"] \arrow[d, "\bigoplus G_k C(T^{CW}(Q(j),\eta_j))"'] & C(\widetilde{B})_k \otimes (C(\hat{F} \times \hat{F}), U \otimes U) \arrow[d, "\bigoplus G_k C(T^{CW}(\Delta^B \circ Q(j),\eta_j \times \eta_j))"] \\
G_k C(X) \arrow[r, "G_k C(\Delta^X)"] & G_k C(X \times X)
\end{tikzcd} \tag{3.3.5}$$

By equation 3.3.3 we know that

$$\bigoplus_{j \in J_k} G_k C(T^{CW}(Q(j), \eta_j)) = (\mathcal{E}_k^X)^{-1}$$

and by proposition 3.16 to the given CW-model for $\widetilde{E} \times \widetilde{E}$ (equation 3.3.4) we deduce that

$$\begin{aligned}
&\bigoplus_{j \in J_k} G_k C(T^{CW}(\Delta^b \circ Q(j), \eta_j \times \eta_j)) \\
&\quad = ((\theta^{C(\widetilde{B}),C(\widetilde{B})})^{-1} \circ (\mathcal{E}_k^X \otimes_{\mathbf{Z}} \mathcal{E}_k^X) \circ \theta^{X,X})^{-1} \\
&\qquad \circ \bigoplus_{j \in J_k} \bar{C}(\Delta^B \circ Q(j), \Delta^B \circ q(j), \eta_j \times \eta_j) \otimes (C(\hat{F}) \otimes_{\mathbf{Z}} C(\hat{F})), U \otimes_{\mathbf{Z}} U) \\
&\quad = (\theta^{X,X})^{-1} \circ ((\mathcal{E}_k^X)^{-1} \otimes_{\mathbf{Z}} (\mathcal{E}_k^X)^{-1}) \circ \theta^{C(\widetilde{B}),C(\widetilde{B})} \\
&\qquad \circ (C(\Delta^{\widetilde{B}})_k \otimes (C(\hat{F}) \otimes_{\mathbf{Z}} C(\hat{F})), U \otimes_{\mathbf{Z}} U)) \\
&\qquad : C(\widetilde{B})_k \otimes (C(\hat{F} \times \hat{F}), U \otimes U) \to G_k C(X \times X)
\end{aligned}$$

If we now substitute these two expressions for the vertical maps of diagram 3.3.5 and do a significant amount of rearranging to make it humanly readable we obtain the required diagram (equation 3.2.2):

$$\begin{tikzcd}
G_* C(X) \arrow[r, "\mathcal{E}"] \arrow[d, "G_* C(\Delta^X)"] & C(\widetilde{B}) \otimes (C(\hat{F}), U) \arrow[d, "G_* C(\Delta^{\widetilde{B}}) \otimes \Delta^{\hat{F}}"] \\
G_*(C(X) \otimes C(X)) \arrow[d, "\theta^{X,X}"'] & (C(\widetilde{B}) \otimes C(\widetilde{B})) \otimes (C(\hat{F}) \otimes C(\hat{F}), U \otimes U) \arrow[d, "\theta^{C(\widetilde{B}),C(\widetilde{B})}"] \\
G_* C(X) \otimes G_* C(X) \arrow[r, "\mathcal{E}^X \otimes \mathcal{E}^X"] & (C(\widetilde{B}) \otimes (C(\hat{F}), U)) \otimes (C(\widetilde{B}) \otimes (C(\hat{F}), U))
\end{tikzcd} \tag{3.3.6}$$

□

Chapter 4

The signature of a fibration.

In this chapter we will show how the signature of a fibration may be obtained from the symmetric complex of the base space and the action of $\pi_1(B)$ on the middle dimension of the fibre. We present a new way of obtaining this result from our description of a fibration.

4.1 Compatible orientations.

We say a fibration is Poincaré if F and B are Poincaré spaces of dimension m and n respectively and if the action of $\pi_1(B)$ on $H_m(F)$ the top dimensional homology of F is trivial. It is already known [Got79] that in this case E is a Poincaré space. We wish to describe explicitly how an orientation of E is determined from that of F and B; we refer to this as the *compatible* orientation.

One way of doing this is via spectral sequences. For a fibration satisfying these conditions we know that

$$H_{n+m}(E) = E^2_{n,m}(E) = H_n(B; H_m(F)) = H_n(B) \otimes H_m(F)$$

so the natural orientation of E is $[B] \otimes [F]$ under this identification. We wish to describe this using the algebraic model of a fibration developed in the previous chapter. By Theorem 3.11 there exists a filtered $\pi_1(E)$-CW-complex X which is filtered homotopy equivalent to $\widetilde{E}$ and such that there exists a natural isomorphism:

$$\mathcal{E} : G_*C(X) \to C(\widetilde{B}) \otimes (C(\hat{F}), U)$$

We write $(C(F), U)$ for the representation of $\mathbf{Z}[\pi_1 B]$ in $\mathbb{D}(\mathbf{Z})$ given by the action of $\pi_1(B)$ on $C(F)$. After tensoring with $\mathbf{Z}$ we have a chain isomorphism

$$\mathcal{E} \otimes_{\mathbf{Z}[\pi_1(E)]} \mathbf{Z} : G_*C(X) \otimes_{\mathbf{Z}[\pi_1(E)]} \mathbf{Z} \to C(\widetilde{B}) \otimes (C(F), U)$$

in the category $\mathbb{D}(\mathbf{Z})$. From now on for the sake of clarity we'll write $C(\widetilde{E})$ for $C(X)$ and $C(E)$ for $C(X) \otimes_{\mathbf{Z}[\pi_1(E)]} \mathbf{Z}$.

Definition 4.1. *We define an (m, n)-cycle x of a chain complex G_*C in $\mathbb{D}(R)$ to be a cycle in G_nC_m such that $G_*d(x)$ represents 0 in $H_m(G_{n-1}C)$.*

Let f be a cycle in $C(F)_m$ representing $[F]$ and b a cycle in $C(B)_n$ representing $[B]$. The tensor product $b \otimes f$ is a well-defined m-cycle of $C_n(B) \otimes (C(F), U)$. The fact that $\pi_1(B)$ acts trivially on $H_m(F)$ implies that $d^B \otimes (C(F), U)$ applied to $b \otimes f$ represents zero in $H_m(G_{n-1}C(E))$. Hence $b \otimes f$ is a (m, n)-cycle. If we now take the inverse image of $b \otimes f$ under (some chain-level representative of) the chain isomorphism $\mathcal{E}^{-1} \otimes_{\mathbf{Z}[\pi_1(B)]} \mathbf{Z}$ we get a well-defined (m, n)-cycle in $G_nC(X)_m$ which we denote by e. But this may also be considered to be an element of $C(E)_{n+m}$ and because it is in the top dimension it is a cycle in $C(E)_{n+m}$. It is the class of this cycle which defines the compatible orientation of E which we denote by $[E]$.

4.2 Duality.

4.2.1 Duality for filtered complexes.

We now work over an additive category with involution $\mathbb{A}$. We recall from [HKR05]:

Definition 4.2.

- *We define the n-*filtered dual *M^* of a filtered object M of filtration degree at most m to be the filtered object with filtration quotients:*

$$f_jM^* = M_n^* \oplus M_{n-1}^* \oplus ... \oplus M_{n-j}^*$$

Observe that if $f : M \to N$ is a filtered map of filtered objects then $f^* : N^* \to M^*$ is a filtered map with respect to this filtration.

- *Given a filtered chain complex F_*C of dimension $n+m$ and with the degree of each F_rC not exceeding n we define the (n, m)-*filtered dual complex *$F_*^{n,m}C$ to be the filtered chain complex with objects $F_r^{n,m}C$ the m-dual of $F_{n+m-r}C$, and differentials as one would expect but with a rather unwieldy looking sign term; specifically:*

$$d_j^{n,m} = (-)^{r+s+j(m+r)} d_j^* : F_r^{n,m}C_s = C^*_{n+m-r,n-s} \to C^*_{n+m-r+1,n-s+j}$$

The point of this sign term will be revealed in the next lemma. In [HKR05] $F_*^{n,m}C$ is denoted by $F_*^{dual}C$.

- *The derived category with m-involution $\mathbb{D}_m(\mathbb{A})$ is the category $\mathbb{D}(\mathbb{A})$ equipped with the involution:*

$$* : C \mapsto C^{m-*}$$

and for morphisms $f : C \to D$

$$*f = f^{m-*} : D^{m-*} \to C^{m-*}$$

(see sign convention 2.3). The natural equivalence $e_{\mathbb{D}(\mathbb{A})}(A) : A \to **A$ for any object A is given by:

$$e_{\mathbb{D}(\mathbb{A})}(A) = (-)^{mr} e_{\mathbb{A}} : A_r \to **A_r$$

Lemma 4.3 ([HKR05] lemma 12.23). *The associated complex of the (n, m)-filtered dual of a filtered complex C satisfies:*

$$G_*(F_*^{n,m}C) = (G_*C)^{n-*}$$

when G_*C is considered to lie in the category $\mathbb{D}_m(\mathbb{A})$.

We have a chain equivalence ([HKR05] lemma 12.25):

$$\theta_{F_*C} : F_*^{n,m}C \to C^{n+m-*} \tag{4.2.1}$$

given by the direct sum of the following map on the filtration quotients:

$$(-)^{s(m+r+1)} : C^*_{n+m-r,n-s} \to C^*_{n+m-r,n-s}$$

4.2.2 Duality for the transfer functor.

In definition 3.10 we introduced (following [LR88]) the notion of a transfer functor $F : \mathbb{A}(R)^{based} \to \mathbb{D}(\mathbb{A})$. We now extend this (still following [LR88]) to a functor of additive categories with involution:

Definition 4.4.

- *A non-singular symmetric form (A, α) in an additive category with involution $\mathbb{A}$ consists of an object $A \in \mathbb{A}$ and an isomorphism $\alpha : A^* \to A$ such that $\alpha^* = \alpha$.*

- *Given a non-singular symmetric form (A, α) we define the ring with involution $\mathrm{Hom}^{\alpha}_{\mathbb{A}}(A, A)^{op}$ to have underlying ring $\mathrm{Hom}_{\mathbb{A}}(A, A)^{op}$ and involution given by:*

  $$*(F : A \to A) = \alpha f^* \alpha^{-1}$$

  A symmetric representation *(A, α, U) of a ring with involution R in an additive category with involution $\mathbb{A}$ consists of a non-singular symmetric form (A, α) in $\mathbb{A}$ along with a morphism of rings with involution:*

  $$R \to \mathrm{Hom}^{\alpha}_{\mathbb{A}}(A, A)^{op}$$

  In particular, (A, U) is a representation of R (definition 3.10).

- *Given a symmetric representation (A, α, U) we define the* transfer functor $F : \mathbb{A}^{based}(R) \to \mathbb{A}$*, a functor of categories with involution, to be the functor of definition 3.10 determined by (A, U). To be a functor of categories with involution we define the natural equivalence $H : *F \to F* : \mathbb{A}(R) \to \mathbb{A}$ to be:*

$$H(R^n) = \bigoplus_n \alpha : *F(R^n) = \bigoplus_n A^* \to F(R^n) = \bigoplus_n A$$

Suppose we have a chain map $f : C^{m-*} \to D$ of chain complexes in a category $\mathbb{A}$. Then we can regard f as a morphism in $\mathbb{D}_m(\mathbb{A})$. When we apply the involution of $\mathbb{D}_m(\mathbb{A})$ we get the morphism

$$f^{m-*} : D^{m-*} \to (C^{m-*})^{m-*}$$

When we compose this with the inverse of the natural equivalence $e_{\mathbb{D}_m(\mathbb{A})}(C)$ we get:

$$e_{\mathbb{D}_m(\mathbb{A})}(C) \circ f^{m-*} = Tf : D^{m-*} \to C$$

Therefore if (C, ϕ) is an m-dimensional symmetric Poincaré complex in $\mathbb{A}$ the pair (C, ϕ_0) define a non-singular symmetric form in $\mathbb{D}_m(\mathbb{A})$.

The main example of a symmetric representation comes from a fibration $F^m \to E \to B$ satisfying assumption 1.2 where F is an m-dimensional Poincaré space. We choose some CW-complex F^{CW} homotopy equivalent to F and apply the symmetric construction of section 2.2 to obtain a symmetric Poincaré complex $(C(\hat{F}), \phi^F)$. For each $g \in \pi_1(B)$ the map $U(g) : \hat{F} \to \hat{F}$ is a homotopy equivalence so the induced maps $U(g) : C(\hat{F}) \to C(\hat{F})$ satisfy $U(g)\phi_0^F U(g)^{n-*} = \phi_0^F : C(\hat{F})^{n-*} \to C(\hat{F})$. Hence $\phi_0^F U(r) = U(r^*)\phi_0^F$ so the triple $(C(\hat{F}), \phi_0^F, U)$ is a symmetric representation of $\mathbf{Z}[\pi_1(B)]$ in the category $\mathbb{D}_n(\mathbf{Z}[\pi_1(E)])$. We refer to this as the symmetric representation *associated* to F.

We can now state one of the main theorems of this section which describes the symmetric Poincaré complex of the total space of a fibration.

Theorem 4.5. *Let $F^m \to E^{m+n} \xrightarrow{p} B^n$ be a Poincaré fibration satisfying assumption 1.2 and $(C(\hat{F}), \phi_0^F, U)$ the symmetric representation associated to p defined above. Let $(C(\widetilde{B}), \phi^B)$ be an n-dimensional symmetric Poincaré complex representing $\widetilde{B}$. Then there exists an $(n+m)$-dimensional symmetric Poincaré complex $(C(\widetilde{E}), \phi^E)$ representing $\widetilde{E}$ satisfying:*

1. *The complex $C(\widetilde{E})$ is n-filtered and there exists a chain isomorphism (in $\mathbb{D}_m(\mathbb{A})$):*

$$\mathcal{E} : G_* C(\widetilde{E}) \to C(\widetilde{B}) \otimes (C(\hat{F}), \phi_0^F, U)$$

2. *The chain map* $\phi_0^E \circ \theta_{F_*C(\widetilde{E})} : F^{n,m}C(\widetilde{E}) \to C(\widetilde{E})$ *is a filtered map and satisfies:*

$$\begin{aligned}\mathcal{E} \circ G_*(\phi_0^E \circ \theta_{F_*C(\widetilde{E})}) \circ \mathcal{E}^{n-*} &= \phi_0^B \otimes (C(\hat{F}), \phi_0^F, U) \circ H(C(\widetilde{B})) \\ &: (C(\widetilde{B}) \otimes (C(\hat{F}, \phi^F, U)))^{n-*} \to C(\widetilde{B}) \otimes (C(\hat{F}), \phi^F, U)\end{aligned}$$

Proof. Let $\Delta^B : C(B) \to C(\widetilde{B}) \otimes_{\mathbf{Z}[\pi_1(B)]} C(\widetilde{B})$ be the chain diagonal approximation such that

$$\phi_0^B = \backslash(\Delta^B([B])) : C(\widetilde{B})^{n-*} \to C(\widetilde{B})$$

. By theorem 3.11 we can find a $\pi_1(E)$-space filtered homotopy equivalent to $\widetilde{E}$ (which we now replace the space $\widetilde{E}$ with), a filtered chain diagonal approximation $\Delta^E : C(E) \to C(\widetilde{E}) \otimes_{\mathbf{Z}[\pi_1(E)]} C(\widetilde{E})$ and a chain isomorphism $\mathcal{E} : G_*C(\widetilde{E}) \to C(\widetilde{B}) \otimes (C(\hat{F}), U)$ such that the diagram:

$$\begin{tikzcd}
G_*C(E) \arrow[rr, "\mathcal{E}\otimes_{\mathbf{Z}[\pi_1(E)]}\mathbf{Z}"] \arrow[d, "G_*C(\Delta^E)"] & & C(\widetilde{B}) \otimes (C(F), U) \arrow[d, "C(\Delta^{\widetilde{B}})\otimes\Delta^{\hat{F}}"] \\
G_*(C(\widetilde{E}) \otimes_{\mathbf{Z}[\pi_1(E)]} C(\widetilde{E})) \arrow[d, "\theta^{E,E}"'] & & (C(\widetilde{B}) \otimes_{\mathbf{Z}[\pi_1 B]} C(\widetilde{B})) \otimes (C(\hat{F}) \otimes_{\mathbf{Z}[\pi_1(E)]} C(\hat{F}), U \otimes U) \arrow[d, "\theta^{C(\widetilde{B}),C(\widetilde{B})}"] \\
G_*C(\widetilde{E}) \otimes_{\mathbf{Z}[\pi_1(E)]} G_*C(\widetilde{E}) \arrow[rr, "\mathcal{E}\otimes\mathcal{E}"] & & (C(\widetilde{B}) \otimes (C(\hat{F}), U)) \otimes_{\mathbf{Z}[\pi_1(E)]} (C(\widetilde{B}) \otimes (C(\hat{F}), U))
\end{tikzcd} \tag{4.2.2}$$

commutes (we are now considering $\mathbf{Z}[\pi_1(E)]$ as a ring with involution so these tensor products are well-defined). We have an orientation $[E]$ for E which corresponds to $[B] \otimes [F]$ under the isomorphism $\mathcal{E} \otimes \mathbf{Z}$ and we apply the symmetric construction to form the symmetric Poincaré complex $(C(\widetilde{E}), \phi^E)$ with $\phi_0^E = \backslash(\Delta^E([E]))$. To avoid confusion we will write $G_*[E]$ when the orientation in considered to lie in $G_nC(E)_m$ and $[E]$ when considered to lie in $C(E)_{n+m}$.

For any chain complexes G_*C and G_*D in $\mathbb{D}_m(R)$ a (m,n)-chain x in $G_*C \otimes_R G_*D$ determines a chain map

$$\backslash x : (G_*C)^{n-*} \to G_*D$$

which is defined on each component

$$(G_{n-r}C)^{m-s} \to (G_rD)_s$$

by the usual slant product on the component of x contained in $(G_{n-r}C)_{m-s} \otimes_{\mathbf{Z}} (G_rD)_s$. This map is determined up to chain homotopy by the class of x in $H_m((G_*C \otimes G_*D)_n)$.

We now apply this to the situation of a fibration. One can easily see from the definition of the slant product and the fact that Δ^X is a filtered map that the composition

$$\phi_0^E \circ \theta_{F_*C(\widetilde{E})} = \backslash(\Delta^E([E])) \circ \theta_{F_*C(\widetilde{E})} : F^{n,m}(C(\widetilde{E})) \to C(\widetilde{E})$$

is a filtered map and that moreover

$$G_*(\phi_0^E \circ \theta_{F_*C(\widetilde{E})}) = \backslash(\theta^{E,E} \circ G_*\Delta^E(G_*[E])) : G_*(C(\widetilde{E}))^{n-*} \to G_*C(\widetilde{E})$$

By the commutativity of diagram 4.2.2 we see that:

$$\begin{aligned}
\mathcal{E} \circ G_*(\phi_0^E \circ \theta_{F_*C(\widetilde{E})}) \circ \mathcal{E}^{n-*} &= \mathcal{E} \circ \backslash(\theta^{E,E} \circ G_*\Delta^X(G_*[E])) \circ \mathcal{E}^{n-*} \\
&= \backslash(\theta^{C(\widetilde{B}),C(\widetilde{B})} \circ G_*C(\Delta^{\widetilde{B}}) \otimes \Delta^{\hat{F}} \circ \mathcal{E}(G_*[E])) \\
&= \backslash(\theta^{C(\widetilde{B}),C(\widetilde{B})} \circ G_*C(\Delta^{\widetilde{B}}) \otimes \Delta^{\hat{F}}([B] \otimes [F])) \\
&= \phi_0^B \otimes (C(\hat{F}), \phi_0^F, U) \\
&= \phi_0^B \otimes (C(\hat{F}), \phi_0^F, U) \circ H(C(\widetilde{B}))
\end{aligned}$$

as required. □

Question 4.6. *How much information does this give us about the symmetric signature of $\widetilde{E}$?*

4.3 The signature of a fibre bundle.

We will now describe how the signature of a fibre bundle is determined by the symmetric structure of the base space and the action of $\pi_1(B)$ on the middle-dimensional homology of the base space.

Definition 4.7. *A* $(\mathbf{Z}, m)$-symmetric representation (A, α, U) *of a group ring* $\mathbf{Z}[\pi]$ *is a symmetric representation of* $\mathbf{Z}[\pi]$ *in the category* $\mathbb{A}(\mathbf{Z}^{(m)})$*, where* $\mathbf{Z}^{(m)}$ *is the ring* $\mathbf{Z}$ *with involution given by* $a^* = (-)^m a$.

In other words the form α is $(-)^m$-symmetric.

The most important examples of $(\mathbf{Z}, m)$-symmetric representations are those constructed from fibre bundles: Let $F^{2n} \to E \to B$ be a fibration satisfying assumption 1.2 with F a $2m$-dimensional Poincaré space. Cup product on the fibre F gives us a $(-)^m$-symmetric form

$$\phi^F : H^m(F, \mathbf{Z})/torsion \to H_m(F, \mathbf{Z})/torsion$$

We write $K = H_m(F, \mathbf{Z})/torsion$ and identify $H^m(F, \mathbf{Z})/torsion \cong K^*$ via the universal coefficient theorem. The action of $\pi_1 B$ on the fibre determines a unitary representation

$$U : \pi_1 B \to \mathrm{Aut}(K, \phi^F)^{op}$$

which extends in the obvious way to a ring representation

$$U : \mathbf{Z}[\pi_1 B] \to \mathrm{Hom}(K, K)^{op}$$

satisfying the required property. The triple (K, ϕ^F, U) is a $(\mathbf{Z}, m)$-symmetric representation of $\mathbf{Z}[\pi_1 B]$; we refer to it as the $(\mathbf{Z}, m)$-symmetric representation associated to the fibration $F^{2m} \to E \to B$.

Definition 4.8. *Let (C, ϕ) be an n-dimensional symmetric Poincaré complex over $\mathbf{Z}[\pi]$ and (A, α, U) an $(\mathbf{Z}, m)$-symmetric representation of $\mathbf{Z}[\pi]$. We define the* twisted product *$(A, \alpha, U) \otimes_R (C, \phi)$ to be the $(2m+n)$-dimensional symmetric complex over $\mathbf{Z}$ with chain complex $S^m C \otimes (A, \alpha, U)$ and morphisms $\phi \otimes (A, \alpha, U)_s$ given by:*

$$\phi \otimes (A, \alpha, U)_s = \phi_s \otimes (A, \alpha, U) \circ H(C_{n+s+m-r}) : C^{n+s+m-r} \otimes (A, \alpha, U) \to C_{m+r}$$

What we're doing here is applying the functor $-\otimes(A, \alpha, U)$ and then bumping up the dimension to make it a symmetric complex over $\mathbf{Z}$ with the standard involution. It is simply a matter of sign counting to check that this is a $(2m+n)$-dimensional symmetric complex. Observe that $(C, \phi) \otimes (A, \alpha, U)$ is Poincaré if and only if (C, ϕ) is Poincaré. We give a simple example:

Let (A, α) be a $(-)^m$-symmetric form. Then we can form a $(\mathbf{Z}, m)$-symmetric form (A, α, ϵ) where the representation

$$\epsilon : \mathbf{Z}[\pi] \to \mathrm{Hom}(A, A)^{op}$$

is given by $\epsilon(r) = I\epsilon(r) : A \to A$. In this case the functor

$$- \otimes (A, \alpha, \epsilon) : \mathbb{A}^{based} \to \mathbf{Z}^{(m)}$$

is simply the usual tensor product with A. If we now consider (A, α) to be a an element of $L^{2m}(\mathbf{Z})$ then the twisted product of an n-dimensional symmetric Poincaré complex (C, ϕ) with (A, α) is equal to the usual product

$$L^n(\mathbf{Z}) \otimes L^{2m}(\mathbf{Z}) \to L^{n+2m}(\mathbf{Z})$$

defined by Ranicki [Ran80a]. We refer to this as the *untwisted product* and write it as $(C, \phi) \otimes (A, \alpha)$. It is a result of [Ran80a] that:

$$\mathrm{sign}((C, \phi) \otimes (A, \alpha)) = \mathrm{sign}(C, \phi)\mathrm{sign}(A, \alpha)$$

Our main result in this chapter is the following theorem:

Theorem 4.9. *Let $F^{2m} \to E^{2n+2m} \to B^{2n}$ be a Poincaré fibration satisfying assumption 1.2, let $(C(\widetilde{B}), \phi^B)$ be a symmetric Poincaré complex representing B and denote by (K, ϕ^F, U) the $(\mathbf{Z}, m)$-symmetric representation associated to fibration. Then the signature of E is equal to the signature of the symmetric Poincaré complex $(C(\widetilde{B}), \phi^B) \otimes (K, \phi^F, U)$.*

We will now work towards proving this result. One could equally well prove it using an argument based on the Serre spectral sequence. The proof we give here is based on our description of the symmetric complex of a fibration.

4.4 A signature for the derived category.

In this section we will construct a signature for $2n$-dimensional symmetric chain equivalences (G_*C, ϕ) in $\mathbb{D}_{2m}(\mathbf{Z})$ with $n+m$ even which we denote by $\mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}(G_*C, \phi)$. This signature will induce a well-defined map:

$$\mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})} : L^{2n}(\mathbb{D}_{2m}(\mathbf{Z})) \to \mathbf{Z}$$

We construct $\mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}(G_*C, \phi)$ as follows: First of all we may form a symmetric chain equivalence in $\mathbb{D}_{2m}(\mathbf{R})$ by tensoring (G_*C, ϕ) with the reals (to avoid extra notation we'll still call this complex (G_*C, ϕ)). We now define a new $2n$-dimensional symmetric chain equivalence $(H_*C, H_*\phi)$ in $\mathbb{D}_{2m}(\mathbf{Z})$ with H_rC the $r'th$ object in H_*C give by:

$$H_rC_s = H_s(G_rC)$$

with differentials $H_rd : H_rC \to H_{r-1}C$ and the map $H_*\phi$ the maps induced by G_*d and ϕ respectively on homology. We now define yet another new pair $(H'_*C; H'_*\phi)$ by:

$$H'_rC = H_rC/\mathrm{Im}(H_*d : H_{r+1}C \to H_rC)$$

with trivial differentials and with $H'_*\phi$ the map induced by $H_*\phi$ (this is well-defined on the quotients). The process of going from (G_*C, ϕ) to $(H'_*C; H'_*\phi)$ is rather similar to forming the E^2-term of a spectral sequence from the E^0-term. Observe that G_*C (over $\mathbf{R}$) is chain equivalent to H'_*C so this symmetric chain equivalence in $\mathbb{D}_{2m}(\mathbf{Z})$ is equivalent to the one we started with. The map $H_n\phi : (H_nC_m)^* \to H_nC_m$ in the middle dimension is a symmetric form over $\mathbf{R}$; we define the signature $\mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}(G_*C, \phi)$ to be the signature of this form.

Lemma 4.10. *The map $\mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})} : L^{2n}(\mathbb{D}_{2m}(\mathbf{Z})) \to \mathbf{Z}$ is well-defined, that is if (G_*C, ϕ) and (G_*C', ϕ') represent the same element of $L^{2n}(\mathbb{D}_{2m}(\mathbf{Z}))$ then*

$$\mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}(G_*C, \phi) = \mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}(G_*C', \phi')$$

Proof. The construction of $\mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}$ only depends on the homotopy class of (G_*C,ϕ); therefore it is sufficient to show that $\mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}(G_*C,\phi) = 0$ for elements representing zero in $L^{2n}(\mathbb{D}_{2m}(\mathbf{Z}))$ since $\mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}$ is clearly additive. Let (G_*C,ϕ) be such an element and $(f : G_*C \to G_*D, (\delta\phi,\phi))$ a suitable null-cobordism. We can play the same game to form a null-cobordism $(H'_*f : H'_*C \to H'_*D, (H'_*\delta\phi, H'_*\phi))$ of $(H'_*C, H'_*\phi)$ with the differentials in H'_*D zero.

We know that

$$\begin{pmatrix} H'_*\delta\phi_0 & H'_*fH'_*\phi_0 \end{pmatrix} : \mathcal{C}(H'_*f)^{2n+1-*} \to H'_*D$$

is a chain equivalence (of chain complexes in $\mathbb{D}(\mathbf{R})$). The chain complex $\mathcal{C}(H'_*f)^{2n+1-*}$ is chain equivalent to the chain complex $H'_*\mathcal{C}(f)$ (in $\mathbb{D}(\mathbf{R})$) given by:

$$H_r\mathcal{C}(f)_s = \mathrm{Ker}((H'_{2n+1-r}f_{2m-s})^*) \oplus \mathrm{Coker}((H'_{2n-r}f_{2m-s})^*)$$

Hence the induced map

$$\begin{pmatrix} H'_*\delta\phi_0 & H'_*fH'_*\phi_0 \end{pmatrix} : \mathrm{Ker}((H'_{2n+1-r}f_{2m-s})^*) \oplus \mathrm{Coker}((H'_{2n-r}f_{2m-s})^*) \to H'_rD_s$$

is an isomorphism of $\mathbf{R}$-vector spaces (observing that $f\phi_0f^* = 0$ implies that $f\phi_0f$ is well-defined on $\mathrm{Coker}(f^*)$). We have a commutative diagram:

$$\begin{tikzcd}
(H'_nD_m)^* \arrow[r, "(H'_nf)^*"] & (H'_nC_m)^* \arrow[r, "(0\ i)"] \arrow[d, "H'_n\phi"] & \mathrm{Ker}((H'_{n+1}f_m)^*) \oplus \mathrm{Coker}((H'_nf_m)^*) \arrow[d, "(H'_{n+1}\delta\phi_0\ H'_nfH'_n\phi_0)"] \\
& H'_nC_m \arrow[r, "H'_nf"] & H'_nD_m
\end{tikzcd}$$

Both of the vertical maps are isomorphisms of $\mathbf{R}$-vector spaces and the top row is exact. It follows that $\mathrm{Im}(H'_nf)$ is a Lagrangian for the form $H'_n\phi_0 : (H'_nC_m)^* \to H'_nC_m$ so $\mathrm{sign}(C,\phi) = 0$ as required. $\square$

If one could define a map $L^n(\mathbb{D}_m(\mathbf{Z}[\pi_1(E)])) \to L^{n+m}(\mathbf{Z}[\pi_1(E)])$ then one might hope to construct the symmetric transfer map as the composition:

$$L^n(\mathbf{Z}[\pi_1(B)]) \xrightarrow{-\otimes(C(\hat{F}),\phi^F,U)} L^n(\mathbb{D}_m(\mathbf{Z}[\pi_1(E)])) \to L^{n+m}(\mathbf{Z}[\pi_1(E)])$$

and get a description of the symmetric signature of $\widetilde{E}$. This would be directly analogous to the construction of Lück and Ranicki [LR92] of the surgery transfer map. They construct a map $L_n(\mathbb{D}_m(\mathbf{Z}[\pi_1(E)])) \to L_{n+m}(\mathbf{Z}[\pi_1(E)])$ in the quadratic case by taking advantage of the fact that one can perform surgery below the middle dimension on quadratic complexes. The difficulty in constructing this map in the symmetric case is highlighted in [LR92] as the chief problem to be overcome in understanding the symmetric signature of $\widetilde{E}$.

Let (C, ϕ, U) be a symmetric representation of a group ring $\mathbf{Z}[\pi]$ in $\mathbb{D}_{2m}(\mathbf{Z})$. Then we can construct an $(\mathbf{Z}, m)$-symmetric representation $(A, \alpha, \bar{U})$ of π by $A = H_m(C)/torsion$, α the map induced from A^* to A by ϕ_0 and $\bar{U}(\pi) = U(\pi)_m$ the maps induced on A by U. Observe that if $(C(F), \phi^F, U)$ is the symmetric representation associated to a fibration then the $(\mathbf{Z}, m)$-symmetric representation constructed from $(C(F), \phi^F, U)$ in this way in the $(\mathbf{Z}, m)$-symmetric representation associated to the fibration.

We have functors of additive categories with involution:

$$- \otimes (C, \phi, U) : \mathbb{A}^{based} \to \mathbb{D}_{2m}(\mathbf{Z})$$

$$- \otimes (A, \alpha, U) : \mathbb{A}^{based} \to \mathbb{A}(\mathbf{Z}^{(m)})$$

which give rise to maps of symmetric L-groups:

$$- \otimes (C, \phi, U) : L^n(\mathbf{Z}[\pi]) \to L^n(\mathbb{D}_{2m}(\mathbf{Z}))$$

$$- \otimes (A, \alpha, U) : L^n(\mathbf{Z}[\pi]) \to L^{n+2m}(\mathbf{Z})$$

We can relate these two maps:

Lemma 4.11. *Let (C, ϕ, U), $(A, \alpha, \bar{U})$ be as above. Then for each n such that $n \equiv m (\mathrm{mod}\ 2)$ the diagram:*

$$\begin{tikzcd}
L^n(\mathbf{Z}[\pi]) \arrow[r, "-\otimes(C,\phi,U)"] \arrow[d, "-\otimes(A,\alpha,\bar{U})"'] & L^n(\mathbb{D}_{2m}(\mathbf{Z})) \arrow[d, "\mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}"] \\
L^{n+2m}(\mathbf{Z}) \arrow[r, "\mathrm{sign}"] & \mathbf{Z}
\end{tikzcd}$$

commutes.

Proof. We can play the same game as before and tensor with $\mathbf{R}$. The key step is to observe that when we take the homology of $C \otimes \mathbf{R}$ the signature then only depends on the middle dimension of $C \otimes \mathbf{R}$.

$$\begin{aligned}
\mathrm{sign} \circ (- \otimes (C, \phi, U)) &= \mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})} \circ (- \otimes (C \otimes \mathbf{R}, \phi \otimes \mathbf{R}, U \otimes \mathbf{R})) \\
&= \mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{R})} \circ (- \otimes (H_*(C \otimes \mathbf{R}), (\phi \otimes \mathbf{R})_*, (U \otimes \mathbf{R})_*)) \\
&= \mathrm{sign} \circ (- \otimes (A \otimes \mathbf{R}, \alpha \otimes \mathbf{R}, \hat{U} \otimes \mathbf{R})) \\
&= \mathrm{sign} \circ (- \otimes (A, \alpha, \hat{U}))
\end{aligned}$$

□

4.5 Finishing the proof

We now complete the proof of Theorem 4.9. We will require the following technical lemma:

Lemma 4.12. *Let C be a filtered $2n+2m$-dimensional ($n+m$ even) chain complex and $\phi : C^{2n+2m-*} \to C$ a map such that*

1. *$T\phi$ is homotopic to ϕ.*
2. *The composition $\phi \circ \theta_C : F^{2n,2m}C \to C$ is a filtered homotopy equivalence.*
3. *The associated map $G_*(\phi)$ (which we denote by $G_*\phi_0$) is a symmetric chain equivalence in the category $\mathbb{D}_{2m}(\mathbf{Z})$.*

Then the signature of the associated symmetric form $\mathrm{sign}_{\mathbb{D}_{2m}}(G_*C, G_*(\phi \circ \theta_C))$ is equal to the signature of (C, ϕ).

Proof. The argument we use here is essentially that of [CHS57] except that we use slightly different language here. Again we first tensor everything with $\mathbf{R}$. We can perform a spectral-sequence type construction to construct a sequence of filtered chain complexes $F_*C^{(i)}$ and chain maps $\phi^{(i)} : (C^{(i)})^{n+m-*} \to C^{(i)}$. We first set $C^{(0)} = F_*C$, $\phi^{(0)} = \phi$ and inductively define:

$$F_jC_i^{(r+1)} = F_jC_i^{(r)}/\mathrm{Im}(d_r : F_{j+r}C_{i+1}^{(r)} \to F_jC_i^{(r)})$$

Because the differentials in each $F_*C^{(r)}$ are filtered we have well-defined filtered differentials $F_*C_{i+1}^{(r+1)} \to F_*C_i^{(r+1)}$ and well-defined maps $\phi^{(r+1)}$ defined to be $\phi^{(r)}$ on the quotient. Note that at each stage $\phi^{(r)} \circ \theta_{C^{(r)}}$ is a filtered map. The associated complex of $(F_*C^{(2)}, \phi^{(2)})$ is $(H'_*C, H'_*\phi)$ so we deduce that $\mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}(G_*\phi) = \mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}(G_*\phi^{(2)})$. By lemma 4 of [CHS57] (which roughly states that the signature is preserved when moving to the next page of a spectral sequence) we see that $\mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}(G_*\phi^{(r)}) = \mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}(G_*\phi^{(r+1)})$. The sequence $(C^{(r)}, \phi^{(r)})$ converges to a term $(C^{(\infty)}, \phi^{(\infty)})$ and by the usual arguments we know that $\mathrm{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}(G_*\phi^{(\infty)}) = \mathrm{sign}(\phi)$. Putting all this together we see that

$$\mathrm{sign}(\phi) = \mathrm{sign}(G_*\phi^{(\infty)}) = \mathrm{sign}(G_*\phi^{(2)}) = \mathrm{sign}(G_*\phi)$$

as required. □

Proof of Theorem 4.9. By Theorem 4.5 we have a symmetric complex $(C(\widetilde{E}), \phi^E)$ representing $\widetilde{E}$ such that $C(\widetilde{E})$ is a filtered complex, $\phi_0 \circ \theta_{C(\widetilde{E})}$ is a filtered map and the symmetric chain equivalence $(G_*C(\widetilde{E}), G_*\phi_0)$ is isomorphic to the symmetric chain equivalence $(C(\widetilde{B}) \otimes (C(\hat{F}), \phi^F, U), \phi_0^B \otimes (C(\hat{F}), \phi^F, U))$. From the above lemma the signature of ϕ_0^E is equal to the signature of $G_*\phi_0^E$ and hence:

$$\begin{aligned}
\operatorname{sign}(E) &= \operatorname{sign}(\phi_0^E) \\
&= \operatorname{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}(G_*\phi_0^E) \\
&= \operatorname{sign}_{\mathbb{D}_{2m}(\mathbf{Z})}(\phi_0^B \otimes (C(F), \phi^F, U) \circ H(\widetilde{B}))
\end{aligned}$$

However we know from lemma 4.11 that this last signature is equal to the signature of the twisted product of $C(\widetilde{B})$ with (K, ϕ^F, U). The Theorem follows. $\square$

Chapter 5

Absolute Whitehead torsion.

This chapter is devoted to the development of the theory of absolute Whitehead torsion; this is a refinement of the the usual Whitehead torsion (see e.g. Milnor [Mil66]). Absolute torsion invariants lie in the unreduced group $K_1(R)$ rather than the usual $\widetilde{K}_1(R)$; here we develop the theory in the even more general case of an additive category $\mathbb{A}$ (following [Ran85]).

In order to make the formulae in this chapter more concise we introduce the notion of a *signed* chain complex; this is a pair (C, η_C) where C is a chain complex and η_C an element of $K_1^{iso}(\mathbb{A})$. The absolute torsion is then defined for contractible signed complexes and chain equivalences of signed complexes. By defining suitably chosen operations on signed chain complexes (e.g. sum, suspension, dual etc...) we can establish nice properties for the absolute torsion. In section 5.5 we apply the absolute torsion to develop a new invariant of symmetric Poincaré complexes; in section 5.6 we show how this can be used as a replacement for the incorrect absolute torsion in [HRT87]. Finally in section 5.8 we identify the "sign" term in the absolute torsion of a Poincaré complex with more traditional invariants. We will see that the absolute torsion and the Euler characteristic allow us to determine the signature modulo four of a $4k$-dimensional complex. This will be applied in chapter 7 to show that the signature of a fibration is multiplicative modulo four.

The material in this chapter is essentially that contained in the preprint [Kor05].

5.1 Absolute torsion of contractible complexes and chain equivalences.

In this section we introduce the absolute torsion of contractible complexes and chain equivalences and derive their basic properties. This closely follows [Ran85] but without the assumption that the complexes are round ($\chi(C) = 0 \in K_0(\mathbb{A})$);

we also develop the theory in the context of signed chain complexes which we will define in this section.

Let $\mathbb{A}$ be an additive category. Following [Ran85] we define:

Definition 5.1. *1. The* class group $K_0(\mathbb{A})$ *has one generator* $[M]$ *for each object in* $\mathbb{A}$ *and relations:*

(a) $[M] = [M']$ *if* M *is isomorphic to* $[M']$.

(b) $[M \oplus N] = [M] + [N]$ *for objects* M, N *in* $\mathbb{A}$.

2. The isomorphism torsion group $K_1^{iso}(\mathbb{A})$ *has one generator* $\tau^{iso}(f)$ *for each isomorphism* $f : M \to N$ *in* $\mathbb{A}$*, and relations:*

(a) $\tau^{iso}(gf) = \tau^{iso}(f) + \tau^{iso}(g)$ *for isomorphisms* $f : M \to N$, $g : N \to P$

(b) $\tau^{iso}(f \oplus f') = \tau^{iso}(f) + \tau^{iso}(f')$ *for isomorphisms* $f : M \to N$, $f' : M' \to N'$

5.1.1 Sign terms

The traditional torsion invariants are considered to lie in $\widetilde{K}_1^{iso}(\mathbb{A})$, a particular quotient of $K_1^{iso}(\mathbb{A})$ (defined below) in which the torsion of maps such as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : C \oplus D \to D \oplus C$ are trivial. In absolute torsion we must consider such rearrangement maps; to this end we recall from [Ran85] the following notation:

Definition 5.2. *Let* C, D *be free, finitely generated chain complexes in* $\mathbb{A}$.

1. The suspension *of* C *is the chain complex* SC *such that* $SC_r = C_{r-1}$ *and* $SC_0 = 0$

2. The sign *of two objects* $X, Y \in \mathbb{A}$ *is the element*

$$\epsilon(X,Y) := \tau^{iso}\left(\begin{pmatrix} 0 & 1_Y \\ 1_X & 0 \end{pmatrix} : X \oplus Y \to Y \oplus X\right) \in K_1^{iso}(\mathbb{A})$$

The sign only depends on the stable isomorphism classes of M *and* N *and satisfies:*

(a) $\epsilon(M \oplus M', N) = \epsilon(M, N) + \epsilon(M', N)$

(b) $\epsilon(M, N) = -\epsilon(N, M)$

(c) $\epsilon(M, M) = \tau^{iso}(-1 : M \to M)$

We may extend ϵ to a morphism of abelian groups:

$$\epsilon\colon K_0(\mathbb{A}) \otimes K_0(\mathbb{A}) \to K_1^{iso}(\mathbb{A}); ([M],[N]) \mapsto \epsilon(M,N)$$

3. *The* reduced isomorphism torsion group $\widetilde{K}_1^{iso}(\mathbb{A})$ *is the quotient:*

$$\widetilde{K}_1^{iso}(\mathbb{A}) := K_1^{iso}(\mathbb{A})/\mathrm{Im}(\epsilon\colon K_0(\mathbb{A}) \otimes K_0(\mathbb{A}) \to K_1^{iso}(\mathbb{A}))$$

4. *The* intertwining *of C and D is the element defined by:*

$$\beta(C,D) := \sum_{i>j} (\epsilon(C_{2i}, D_{2j}) - \epsilon(C_{2i+1}, D_{2j+1})) \in K_1^{iso}(\mathbb{A})$$

Example 5.3. *The reader may find it useful to keep the following example in mind, as it is the most frequently occurring context.*

Let R be an associative ring with 1 such that $\mathrm{rank}_R(M)$ *is well-defined for f.g free modules M. We define $\mathbb{A}(R)$ to be the category of based f.g. R-modules. In this case the map $K_0(\mathbb{A}(R)) \to \mathbf{Z}$ given by $M \mapsto \dim M$ is an isomorphism. We have a forgetful functor:*

$$K_1^{iso}(\mathbb{A}(R)) \to K_1(R)\ ;\ \tau^{iso}(f) \mapsto \tau(f)$$

mapping elements of $K_1^{iso}(\mathbb{A}(R))$ to the more familiar $K_1(R)$ in the obvious way. In particular

$$\mathrm{Im}(\epsilon\colon K_0(\mathbb{A}(R)) \otimes K_0(\mathbb{A}(R)) \to K_1(R)) = \{\tau(\pm 1)\} = \mathrm{Im}(K_1(\mathbf{Z}) \to K_1(R))$$

justifying the terminology of a "sign" term; the map is given explicitly for modules M and N by:

$$\epsilon(M,N) = \mathrm{rank}_R(M)\mathrm{rank}_R(N)\tau(-1)$$

We will make use of the notation:

$$C_{even} = C_0 \oplus C_2 \oplus C_4 \oplus \dots,$$

$$C_{odd} = C_1 \oplus C_3 \oplus C_5 \oplus \dots$$

and as usual we define the Euler characteristic $\chi(C)$ as:

$$\chi(C) = [C_{even}] - [C_{odd}] \in K_0(\mathbb{A})$$

We also recall from [Ran85] proposition 3.4 the following relationships between the "sign" terms:

Lemma 5.4. *Let C, C', D, D' be finite chain complexes over $\mathbb{A}$. Then*

1. $\beta(C,D) = \tau^{iso}((C \oplus D)_{even} \to C_{even} \oplus D_{even})$
   $- \tau^{iso}((C \oplus D)_{odd} \to C_{odd} \oplus D_{odd})$

2. $\beta(C \oplus C', D) = \beta(C,D) + \beta(C',D)$

3. $\beta(C, D \oplus D') = \beta(C,D) + \beta(C,D')$

4. $\beta(C,D) - \beta(D,C) + \sum(-)^r \epsilon(C_r, D_r) = \epsilon(C_{even}, D_{even}) - \epsilon(C_{odd}, D_{odd})$

5. $\beta(SC, SD) = -\beta(C,D)$

6. $\beta(SC, C) = \epsilon(C_{odd}, C_{even})$

5.1.2 Signed chain complexes.

In order to make to formulae in this chapter more concise we introduce the concept of a *signed* chain complex; this is a chain complex with an associated element in $\mathrm{Im}(\epsilon\colon K_0(\mathbb{A}) \otimes K_0(\mathbb{A}) \to K_1^{iso}(\mathbb{A}))$ which we refer to as the *sign* of the complex. We use this element in the definition of the absolute torsion invariants.

Definition 5.5. 1. *A* signed *chain complex is a pair (C, η_C) where C is a finite chain complex in $\mathbb{A}$ and η_C an element of*

$$\mathrm{Im}(\epsilon\colon K_0(\mathbb{A}) \otimes K_0(\mathbb{A}) \to K_1^{iso}(\mathbb{A}))$$

We will usually suppress mention of η_C denoting such complexes as C.

2. *Given a signed chain complex (C, η_C) we give the suspension of C, SC the sign*

$$\eta_{SC} = -\eta_C$$

3. *We define the* sum *signed chain complex of two signed chain complexes (C, η_C), (D, η_D) as $(C \oplus D, \eta_{C \oplus D})$ where $C \oplus D$ is the usual based sum of two chain complexes and $\eta_{C \oplus D}$ defined by:*

$$\eta_{C \oplus D} = \eta_C + \eta_D - \beta(C,D) + \epsilon(C_{odd}, \chi(D))$$

(it is easily shown that $\eta_{(C \oplus D) \oplus E} = \eta_{C \oplus (D \oplus E)}$)

5.1.3 The absolute torsion of isomorphisms

We now define the absolute torsion of a collection of isomorphisms $\{f_r : C_r \to D_r\}$ between two signed chain complexes. Note that the map f need not be a chain isomorphism (i.e. $fd_C = d_D f$ need not hold). In the case where f is a chain isomorphism the torsion invariant defined here will coincide with the definition of the absolute torsion of chain equivalence given later.

Definition 5.6. *The* absolute torsion *of a collection of isomorphisms $\{f_r : C_r \to D_r\}$ between the chain groups of signed chain complexes C and D is defined as:*

$$\tau_{iso}^{NEW}(f) = \sum_{r=0}^{\infty}(-)^r \tau^{iso}(f_r : C_r \to D_r) - \eta_C + \eta_D \in K_1^{iso}(\mathbb{A})$$

Lemma 5.7. *We have the following properties of the absolute torsion of isomorphisms:*

1. *The absolute torsion of isomorphisms is logarithmic, that is for isomorphisms $f : C \to D$ and $f : D \to E$.*

$$\tau_{iso}^{NEW}(gf) = \tau_{iso}^{NEW}(f) + \tau_{iso}^{NEW}(g)$$

2. *The absolute torsion of isomorphisms is additive, that is for isomorphisms $f : C \to D$ and $f' : C' \to D'$*

$$\tau_{iso}^{NEW}(f \oplus g) = \tau_{iso}^{NEW}(f) + \tau_{iso}^{NEW}(g)$$

3. *The absolute torsion of the rearrangement isomorphism:*

$$C \oplus D \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} D \oplus C$$

is $\epsilon(\chi(C), \chi(D)) \in K_1^{iso}(\mathbb{A})$.

4. *The absolute torsion of the isomorphism:*

$$S(C \oplus D) \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} SC \oplus SD$$

is $\epsilon(\chi(D), \chi(C)) \in K_1^{iso}(\mathbb{A})$.

Proof. Parts 1 and 2 follow straight from the definitions. For part 3

$$\begin{aligned}
\tau^{NEW}_{iso}(C \oplus D \to D \oplus C) &= \sum_{r=0}^{\infty}(-)^r \epsilon(C_r, D_r) - \eta_{C\oplus D} + \eta_{D\oplus C} \\
&= \sum_{r=0}^{\infty}(-)^r \epsilon(C_r, D_r) + \beta(C,D) - \beta(D,C) \\
&\quad -\epsilon(C_{odd}, \chi(D)) + \epsilon(D_{odd}, \chi(C)) \\
&= \epsilon(C_{even}, D_{even}) - \epsilon(C_{odd}, D_{odd}) \\
&\quad -\epsilon(C_{odd}, \chi(D)) + \epsilon(D_{odd}, \chi(C)) \\
&= \epsilon(\chi(C), \chi(D))
\end{aligned}$$

For part 4:

$$\begin{aligned}
\tau^{NEW}_{iso}(S(C \oplus D) \to SC \oplus SD) &= \eta_{SC\oplus SD} - \eta_{S(C\oplus D)} \\
&= -\beta(SC, SD) + \epsilon(C_{even}, \chi(SD)) \\
&\quad -\beta(C,D) + \epsilon(C_{odd}, \chi(D)) \\
&= \epsilon(\chi(D), \chi(C))
\end{aligned}$$

□

5.1.4 The absolute torsion of contractible complexes and short exact sequences.

We recall from [Ran85] the following:

Given a finite contractible chain complex over $\mathbb{A}$

$$C\colon C_n \to \ldots \to C_0$$

and a chain contraction $\Gamma\colon C_r \to C_{r+1}$ we may form the following isomorphism:

$$d + \Gamma = \begin{pmatrix} d & 0 & 0 & \ldots \\ \Gamma & d & 0 & \ldots \\ 0 & \Gamma & d & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix} : C_{odd} = C_1 \oplus C_3 \oplus C_5 \ldots \to C_{even} = C_0 \oplus C_2 \oplus C_4 \ldots$$

The element $\tau^{iso}(d+\Gamma) \in K^{iso}_1(\mathbb{A})$ is independent of the choice of Γ and is denoted $\tau(C)$ (following [Ran85] section 3).

We define the *absolute torsion* of a contractible signed chain complex C as

$$\tau^{NEW}(C) = \tau(C) + \eta_C \in K^{iso}_1(\mathbb{A})$$

Given a short exact sequence of signed chain complexes over $\mathbb{A}$:

$$0 \to C \xrightarrow{i} C'' \xrightarrow{j} C' \to 0$$

we may find a sequence of splitting morphisms $\{k : C'_r \to C''_r | r \geq 0\}$ such that $jk = 1 : C'_r \to C'_r$ $(r \geq 0)$ and each $(i\ k) : C_r \oplus C'_r \to C''_r$ $(r \geq 0)$ is an isomorphism. The torsion of this collection of isomorphisms

$$\tau_{iso}^{NEW}((i\ k) : C_r \oplus C'_r \to C''_r)$$

is independent of the choice of the k_r, so we may define the *absolute torsion of a short exact sequence* as:

$$\tau^{NEW}(C, C'', C'; i, j) = \tau_{iso}^{NEW}((i\ k) : C_r \oplus C'_r \to C''_r)$$

Lemma 5.8. *We have the following properties of the absolute torsion of signed contractible complexes:*

1. *Suppose we have a short exact sequence of contractible signed complexes:*

$$0 \to C \xrightarrow{i} C'' \xrightarrow{j} C' \to 0$$

Then

$$\tau^{NEW}(C'') = \tau^{NEW}(C) + \tau^{NEW}(C') + \tau^{NEW}(C, C'', C'; i, j)$$

2. *Let C, C' be contractible signed complexes. Then:*

$$\tau^{NEW}(C \oplus C') = \tau^{NEW}(C) + \tau^{NEW}(C')$$

Proof. 1. From [Ran85] proposition 3.3 we have that

$$\tau(C'') = \tau(C) + \tau(C') + \sum_{r=0}^{\infty} \tau^{iso}((i\ k) : C_r \oplus C'_r \to C'') + \beta(C, C)$$

for some choice of splitting morphisms $\{k : C'_r \to C''_r | r \geq 0\}$. By the definition of the absolute torsion of a short exact sequence and the definition of the sum torsion (noting that contractible complexes have $\chi(C) = 0 \in K_0(\mathbb{A})$) we get:

$$\begin{aligned} \tau^{NEW}(C, C'', C'; i, j) &= \sum_{r=0}^{\infty} (-)^r \tau^{iso}((i\ k) : C_r \oplus C'_r \to C'') \\ &\quad + \beta(C, C') - \eta_C - \eta_{C'} + \eta_{C''} \end{aligned}$$

By comparing these two formulae and the definition of the absolute torsion of a contractible signed complex, the result follows.

2. Apply the above to $C'' = C \oplus C'$.

□

5.1.5 The absolute torsion of chain equivalences.

We make the algebraic mapping cone (2.2) $\mathcal{C}(f)$ into a signed complex by setting

$$\eta_{\mathcal{C}(f)} = \eta_{D\oplus SC}$$

Lemma 5.9. *The absolute torsion of a chain isomorphism $f : C \to D$ of signed chain complexes satisfies:*

$$\tau^{NEW}_{iso}(f) = \tau^{NEW}(\mathcal{C}(f))$$

Proof. In the case of an isomorphism we may choose the chain contraction for $\mathcal{C}(f)$ to be:

$$\Gamma_{\mathcal{C}(f)} = \begin{pmatrix} 0 & 0 \\ (-)^r f^{-1} & 0 \end{pmatrix} : \mathcal{C}(f)_r \to \mathcal{C}(f)_{r+1}$$

We have a commutative diagram:

$$\begin{tikzcd}
(D_1 \oplus C_0) \oplus (D_3 \oplus C_2) \oplus \ldots \arrow[r, "(d_{\mathcal{C}(f)} + \Gamma_{\mathcal{C}(f)})"] \arrow[d] & D_0 \oplus (D_2 \oplus C_1) \oplus (D_4 \oplus C_3) \oplus \ldots \arrow[d] \\
C_0 \oplus D_1 \oplus C_2 \oplus D_3 \ldots \arrow[r, "\begin{pmatrix} f & d_D & 0 & \ldots \\ 0 & -f^{-1} & d_C & \ldots \\ 0 & 0 & f & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}"] & D_0 \oplus C_1 \oplus D_2 \oplus C_3
\end{tikzcd}$$

The torsion of the upper map is $\tau^{iso}(\mathcal{C}(f))$, the torsion of the lower isomorphism is $\sum_{r=0}^{\infty}(-)^r\tau^{iso}(f_r : C_r \to D_r) + \epsilon(C_{odd}, C_{odd})$ and the difference between the torsions of the downward maps is $\sum_{r=0}^{\infty}(-)^r\epsilon(C_r, C_{r-1})$ (using the fact that $C_r \cong D_r$). Hence

$$\begin{aligned}
\tau^{NEW}(\mathcal{C}(f)) &= \tau^{iso}(\mathcal{C}(f)) + \eta_{\mathcal{C}(f)} \\
&= \sum_{r=0}^{\infty}(-)^r\tau^{iso}(f_r : Cr \to D_r) - \sum_{r=0}^{\infty}(-)^r\epsilon(C_r, C_{r-1}) \\
&\quad -\beta(C, SC) + \epsilon(C_{odd}, \chi(SC)) + \epsilon(C_{odd}, C_{odd}) - \eta_C + \eta_D \\
&= \sum_{r=0}^{\infty}(-)^r\tau^{iso}(f_r : C_r \to D_r) - \eta_C + \eta_D \\
&= \tau^{NEW}_{iso}(f)
\end{aligned}$$

(using the formulae of lemma 5.4) □

We can now give a definition of the absolute torsion of a chain equivalence $f : C \to D$ which coincides with the previous definition in the case when f is a chain isomorphism.

Definition 5.10. *We define the* absolute torsion *of a chain equivalence of signed chain complexes $f : C \to D$ as:*

$$\tau^{NEW}(f) = \tau^{NEW}(\mathcal{C}(f)) \in K_1^{iso}(\mathbb{A})$$

In the case where f is a chain isomorphism the above lemma shows that this definition of the torsion agrees with that given in definition 5.6.

Lemma 5.11. *The absolute torsion of a chain equivalence of chain complexes with torsion $f : C \to D$ is:*

$$\tau^{NEW}(f) = \tau(\mathcal{C}(f)) - \beta(D, SC) - \epsilon(D_{odd}, \chi(C)) + \eta_D - \eta_C \in K_1(\mathbb{A})$$

(c.f. definition of torsion on pages 223 and 226 of [Ran85]. The two definitions coincide if C and D are even and $\eta_C = \eta_D$).

Proof. Simply a matter of unravelling definitions. □

We have the following properties of the torsion of chain equivalences:

Proposition 5.12. *1. Let $f : C \to D$ and $g : D \to E$ be chain equivalences of signed chain complexes in $\mathbb{A}$, then*

$$\tau^{NEW}(gf) = \tau^{NEW}(f) + \tau^{NEW}(g) \in K_1^{iso}(\mathbb{A})$$

2. Suppose $f : C \to D$ is map of contractible signed chain complexes. Then

$$\tau^{NEW}(f) = \tau^{NEW}(D) - \tau^{NEW}(C) \in K_1^{iso}(\mathbb{A})$$

3. The absolute torsion $\tau^{NEW}(f)$ is a chain homotopy invariant of f.

4. Suppose we have a commutative diagram of chain maps as follows where the rows are exact and the vertical maps are chain equivalences:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "a"'] & B \arrow[r, "j"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r] & 0 \end{tikzcd}$$

Then

$$\begin{aligned} \tau^{NEW}(b) &= \tau^{NEW}(a) + \tau^{NEW}(c) - \tau^{NEW}(A, B, C; i, j) \\ &\quad + \tau^{NEW}(A', B', C'; i', j') \in K_1^{iso}(\mathbb{A}) \end{aligned}$$

5. *The torsion of a sum $f \oplus f' : C \oplus C' \to D \oplus D'$ is given by:*

$$\tau^{NEW}(f \oplus f') = \tau^{NEW}(f) + \tau^{NEW}(f') \in K_1^{iso}(\mathbb{A})$$

6. *Suppose we have a short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where C is a contractible complex and f is a chain equivalence. Then

$$\tau^{NEW}(f) = \tau^{NEW}(A, B, C; f, g) + \tau^{NEW}(C)$$

Proof. The proofs of these follow those in [Ran85] propositions 4.2 and 4.4, modified where appropriate.

1. We denote by ΩC the chain complex defined by:

$$d_{\Omega C} = d_C : \Omega C_r = C_{r+1} \to \Omega C_{r-1} = C_r$$

We define a chain map

$$h : \Omega \mathcal{C}(g) \to \mathcal{C}(f)$$

by

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} : \Omega \mathcal{C}(g)_r = E_{r+1} \oplus D_r \to \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

The algebraic mapping cone $\mathcal{C}(h)$ fits into the following short exact sequences:

$$0 \to \mathcal{C}(f) \xrightarrow{i} \mathcal{C}(h) \xrightarrow{j} \mathcal{C}(g) \to 0 \tag{5.1.1}$$

$$0 \to \mathcal{C}(gf) \xrightarrow{i'} \mathcal{C}(h) \xrightarrow{j'} \mathcal{C}(-1_D : D \to D) \to 0 \tag{5.1.2}$$

where

$$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \mathcal{C}(f)_r \to \mathcal{C}(h)_r = \mathcal{C}(f)_r \oplus \mathcal{C}(g)_r$$

$$j = \begin{pmatrix} 0 & 1 \end{pmatrix} : \mathcal{C}(h)_r = \mathcal{C}(f)_r \oplus \mathcal{C}(g)_r \to \mathcal{C}(g)_r$$

$$i' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & f \end{pmatrix} : \mathcal{C}(gf)_r = E_r \oplus SC_r \to \mathcal{C}(h)_r = D_r \oplus SC_r \oplus E_r \oplus SD_r$$

$$j' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -f & 0 & 1 \end{pmatrix} : \mathcal{C}(h)_r = D_r \oplus SC_r \oplus E_r \oplus SD_r \to \mathcal{C}(-1_D)_r = D_r \oplus SD_r$$

Applying lemma 5.8 part 1 to the first short exact sequence (5.1.1) we have

$$\tau^{NEW}(h) = \tau^{NEW}(f) + \tau^{NEW}(g) \tag{5.1.3}$$

Notice that

$$\begin{aligned}
\tau_{iso}^{NEW}((i'\ k')) &= \tau_{iso}^{NEW}\left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & f & 0 & 1 \end{pmatrix} : \mathcal{C}(gf)_r \oplus \mathcal{C}(-1_D)_r\right. \\
&\qquad \left. = E_r \oplus SC_r \oplus D_r \oplus SD_r \to \mathcal{C}(h)_r = D_r \oplus SC_r \oplus E_r \oplus SD_r\right) \\
&= \tau_{iso}^{NEW}(D \oplus SC \to SC \oplus D) \\
&\quad + \tau_{iso}^{NEW}(E \oplus SC \to SC \oplus E) \\
&\quad + \tau_{iso}^{NEW}(D \oplus E \to E \oplus D) \\
&= \epsilon(\chi(D), \chi(D))
\end{aligned}$$

(using the results of lemma 5.7, the fact that $\chi(C) = \chi(D) = \chi(E)$ and that f has no effect on the torsion). We also see that $\tau^{NEW}(\mathcal{C}(-1_D)) = \tau_{iso}^{NEW}(-1_D) = \epsilon(\chi(D), \chi(D))$. Applying these two expressions and lemma 5.8 part 1 to the second exact sequence (5.1.2) we see that

$$\tau^{NEW}(gf) = \tau^{NEW}(h)$$

and comparison with (5.1.3) yields the result.

2. By construction we have $\mathcal{C}(0 \xrightarrow{0} D) = D$ and hence

$$\tau^{NEW}(0 \xrightarrow{0} D) = \tau^{NEW}(D)$$

Applying this and the composition formula (part 1) to the composition

$$0 \xrightarrow{0} C \xrightarrow{f} D$$

yields the result.

3. A chain homotopy

$$g \colon f \simeq f' \colon C \to D$$

gives rise to an isomorphism

$$\begin{pmatrix} 1 & (-)^r g \\ 0 & 1 \end{pmatrix} : \mathcal{C}(f) = D \oplus CS \to \mathcal{C}(f') = D \oplus SC$$

which has trivial torsion. Using part 2

$$0 = \tau^{NEW}(\mathcal{C}(f')) - \tau^{NEW}(\mathcal{C}(f))$$

the result follows.

4. We choose splitting morphisms $\{k : C_r \to B_r | r \geq 0\}$ and $\{k' : C'_r \to B'_r | r \geq 0\}$. We have the following short exact sequence of mapping cones:

$$0 \to \mathcal{C}(a) \xrightarrow{\begin{pmatrix} i' & 0 \\ 0 & i \end{pmatrix}} \mathcal{C}(b) \xrightarrow{\begin{pmatrix} j' & 0 \\ 0 & j \end{pmatrix}} \mathcal{C}(c) \to 0$$

We note that

$$\begin{aligned}
& \tau^{NEW}\left(\mathcal{C}(a), \mathcal{C}(b), \mathcal{C}(c); \begin{pmatrix} i' \\ i \end{pmatrix}; \begin{pmatrix} i' \\ i \end{pmatrix}\right) \\
= \; & \tau^{NEW}_{iso}\left(\begin{pmatrix} i' & 0 & k' & 0 \\ 0 & i & 0 & k \end{pmatrix} : A' \oplus SA \oplus C' \oplus SC \to B' \oplus B\right) \\
= \; & \tau^{NEW}_{iso}\left(\begin{pmatrix} i' & k' & 0 & 0 \\ 0 & 0 & i & k \end{pmatrix} : A' \oplus C' \oplus SA \oplus SC \to B' \oplus B\right) \\
& + \tau^{NEW}_{iso}(SA \oplus C' \to C' \oplus SA) \\
= \; & \tau^{NEW}_{iso}((i\ k) : SA \oplus SC \to SB) + \tau^{NEW}_{iso}((i'\ k')) + \epsilon(\chi(SA), \chi(C')) \\
= \; & \tau^{NEW}_{iso}((i\ k) : S(A \oplus C) \to SB) + \tau^{NEW}_{iso}((i'\ k')) \\
& + \tau^{NEW}_{iso}(SA \oplus SC \to S(A \oplus C)) + \epsilon(\chi(C), \chi(A)) \\
= \; & -\tau^{NEW}_{iso}((i\ k) : A \oplus C \to B) + \tau^{NEW}_{iso}((i'\ k')) \\
& - \epsilon(\chi(C), \chi(A)) + \epsilon(\chi(C), \chi(A)) \\
= \; & -\tau^{NEW}_{iso}((i\ k) : A \oplus C \to B) + \tau^{NEW}_{iso}((i'\ k')) \\
= \; & \tau^{NEW}(A', B', C'; i', j') - \tau^{NEW}(A, B, C; i, j)
\end{aligned}$$

The result now follows from applying lemma 5.8 part 1 to the short exact sequence above.

5. Applying the result for a commutative diagram of short exact sequences (part 4) with $a = f : C \to D$, $c = f' : C' \to D'$ and $b = f \oplus f' : C \oplus C' \to D \oplus D'$ yields the result.

6. We have a commutative diagram with short exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[r, "g"] \arrow[d, "1"'] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r, "1"] & B \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

The result follows by applying part 4 to the above diagram.

□

5.1.6 Applications to topology and examples of use.

Let X be a connected finite CW-complex. We may form the cellular chain complex of the universal cover of X as a complex $C(\tilde{X})$ over the fundamental group ring $\mathbf{Z}[\pi_1 X]$; we may further make $C(\tilde{X})$ into a signed complex with an arbitrary choice of $\eta_{C(\tilde{X})}$. For a cellular homotopy equivalence $f : X \to X$ we have an associated chain equivalence $f_* : C(\tilde{X}) \to C(\tilde{X})$; we can make $C(\tilde{X})$ into a signed chain complex by choosing some $\eta_{C(\tilde{X})}$ and define the torsion of f to be

$$\tau^{NEW}(f) := \tau^{NEW}(f_* : C(\tilde{X}) \to C(\tilde{X})) \in K_1(\mathbf{Z}[\pi_1 X])$$

this is independent of the choice of $\eta_{C(\tilde{X})}$. We now give some examples:

1. The torsion of the identity map of any connected CW-complex is trivial.

2. Let $X = \mathbf{CP}^2$; we choose homogeneous coordinates $(x : y : z)$ and we give X a CW-structure as follows:
   0-cell $(1 : 0 : 0)$
   2-cell $(z_1 : 1 : 0)$
   4-cell $(z_1 : z_2 : 1)$
   Let $f : \mathbf{CP}^2 \to \mathbf{CP}^2$ be the cellular self-homeomorphism given by complex conjugation in all three coordinates, that is:

   $$f : (x : y : z) \mapsto (\bar{x} : \bar{y} : \bar{z})$$

   This map preserves the orientation of the 0-cell and 4-cell, and it reverses the orientation of the 2-cell. Hence $\tau^{NEW}(f) = \tau(-1)$. In corollary 5.28 we show that for any orientation preserving self-homeomorphism g of a simply connected manifold of dimension $4k + 2$, that $\tau^{NEW}(g) = 0$. This example shows that for self-homeomorphism f of a $4k$-dimensional manifold it is possible for $\tau^{NEW}(f) \neq 0$

5.2 The signed derived category.

Chapter 6 will require the use of the signed derived category. In this section we define $\mathbb{SD}(\mathbb{A})$ and prove some basic properties.

Definition 5.13. *The signed derived category $\mathbb{SD}(\mathbb{A})$ is the additive category with objects signed chain complexes in $\mathbb{A}$ and morphisms chain homotopy classes of chain maps between such complexes.*

We write $\mathbb{SD}(R)$ for $\mathbb{SD}(\mathbb{A}(R))$

Proposition 5.14. (i) *The Euler characteristic defines a surjection*

$$\chi \;:\; K_0(\mathbb{SD}(\mathbb{A})) \to K_0(\mathbb{A}) \;;\; [C,\eta_C] \mapsto \chi(C) \;=\; \sum_{r=0}^{\infty}(-)^r[C_r] \;.$$

(ii) *Isomorphism torsion defines a forgetful map*

$$i_* \;:\; K_1^{iso}(\mathbb{SD}(\mathbb{A})) \to K_1^{iso}(\mathbb{A}) \;;\; \tau^{iso}(f) \mapsto [\tau^{NEW}(f)] \;=\; \tau^{NEW}(f)$$

which is a surjection split by the injection

$$K_1^{iso}(\mathbb{A}) \to K_1^{iso}(\mathbb{SD}(\mathbb{A})) \;;\; \tau^{iso}(f : A \to B) \mapsto \tau^{NEW}(f : (A,0) \to (B,0)) \;.$$

(iii) *The diagram*

$$\begin{array}{ccc}
K_0(\mathbb{SD}(\mathbb{A})) \otimes K_0(\mathbb{SD}(\mathbb{A})) & \xrightarrow{\;\epsilon\;} & K_1^{iso}(\mathbb{SD}(\mathbb{A})) \\
\chi\otimes\chi \big\downarrow \cong & & \big\downarrow i_* \\
K_0(\mathbb{A}) \otimes K_0(\mathbb{A}) & \xrightarrow{\qquad\epsilon\qquad} & K_1^{iso}(\mathbb{A})
\end{array}$$

commutes, that is the sign of objects (C,η_C), (D,η_D) *in* $\mathbb{SD}(\mathbb{A})$ *has image*

$$i_*\epsilon((C,\eta_C),(D,\eta_D)) \;=\; \epsilon(\chi(C),\chi(D)) \in K_1^{iso}(\mathbb{A}) \;.$$

Proof. (i) A short exact sequence $0 \to C \to D \to E \to 0$ of finite chain complexes in $\mathbb{A}$ determines a relation

$$[C,\eta_C] - [D,\eta_D] + [E,\eta_E] \;=\; 0 \in K_0(\mathbb{SD}(\mathbb{A}))$$

for any signs η_C, η_D, η_E.

(ii) By construction.

(iii) The sign

$$\begin{aligned}
&\epsilon((C,\eta_C),(D,\eta_D)) \\
&= \tau^{NEW}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : (C,\eta_C) \oplus (D,\eta_D) \to (D,\eta_D) \oplus (C,\eta_C)) \in K_1^{iso}(\mathbb{SD}(\mathbb{A}))
\end{aligned}$$

has image

$$\begin{aligned}
&i_*\epsilon((C,\eta_C),(D,\eta_D)) \\
&\quad = \tau^{NEW}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : (C,\eta_C) \oplus (D,\eta_D) \to (D,\eta_D) \oplus (C,\eta_C)) \\
&\quad = \sum_{r=0}^{\infty}(-)^r \tau^{NEW}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : C_r \oplus D_r \to D_r \oplus C_r) + \eta_{D\oplus C} - \eta_{C\oplus D} \\
&\quad = \sum_{r=0}^{\infty}(-)^r\epsilon(C_r,D_r) - \epsilon(\chi(D),\chi(C)) + \sum_{r=0}^{\infty}(-)^r\epsilon(D_r,C_r) \\
&\quad = \epsilon(\chi(D),\chi(C)) \;=\; \epsilon(\chi(C),\chi(D)) \in K_1^{iso}(\mathbb{A}) \;.
\end{aligned}$$

□

5.3 Duality properties of absolute torsion.

In this section we extend the notion of absolute torsion to encompass dual objects and dual maps. We now work over an additive category with involution and introduce the notion of a dual signed complex C^{n-*} (defined below). We prove the following result:

Proposition 5.15. *1. Let C be a contractible signed complex. Then*

$$\tau^{NEW}(C^{n-*}) = (-)^{n+1}\tau^{NEW}(C)^* \in K_1^{iso}(\mathbb{A})$$

2. Let $f : C \to D$ be a chain equivalence of signed chain complexes. Then

$$\tau^{NEW}(f^{n-*} : C^{n-*} \to D^{n-*}) = (-)^n\tau^{NEW}(f)^* \in K_1^{iso}(\mathbb{A})$$

3. Let $f : C^{n-*} \to D$ be a chain equivalence of signed chain complexes. Then the chain equivalence $Tf : D^{n-*} \to C$ (defined below) satisfies:

$$\tau^{NEW}(Tf : D^{n-*} \to C) = (-)^n\tau^{NEW}(f)^* + \frac{n}{2}(n+1)\epsilon(\chi(C), \chi(C)) \in K_1^{iso}(\mathbb{A})$$

The rest of this section will be concerned with defining these concepts and proving proposition 5.15.

Throughout the rest of this chapter $\mathbb{A}$ is an additive category with involution. An involution on $\mathbb{A}$ induces an involution on $K_1^{iso}(\mathbb{A})$ in the obvious way.

We define the sign term

$$\alpha_n(C) = \sum_{r\equiv n+2, n+3 \pmod 4} \epsilon(C^r, C^r) \in K_1^{iso}(\mathbb{A})$$

Given a signed chain complex (C, η_C) we define the *dual signed chain complex with* $(C^{n-*}, \eta_{C^{n-*}})$ by

$$\eta_{C^{n-*}} := (-)^{n+1}\eta_C^* + (-)^{n+1}\beta(C, C)^* + \alpha_n(C) \in K_1^{iso}(\mathbb{A})$$

Lemma 5.16. *Let A and B be elements of $\mathbb{A}$ and C and D chain complexes over $\mathbb{A}$. We have the following basic properties of the absolute torsion in an additive category with involution:*

1. $\chi(C^{n-*}) = (-)^n\chi(C)^*$

2. $\epsilon(A^*, B^*) = \epsilon(B, A)^*$

3. $\beta(C^{n-*}, D^{n-*}) = (-)^n \beta(D, C)^*$

4. For chain isomorphisms $f : C \to D$ *we have that:*

$$\tau^{NEW}(f^{n-*} : D^{n-*} \to C^{n-*}) = (-)^n \tau^{NEW}(f)^* \in K_1^{iso}(\mathbb{A})$$

5.

$$\tau^{NEW}((C \oplus D)^{n-*} \to C^{n-*} \oplus D^{n-*}) = \begin{cases} 0 \\ \epsilon(\chi(C), \chi(D))^* \end{cases} \text{for } \begin{cases} n \text{ even} \\ n \text{ odd} \end{cases}$$

6. $\tau^{NEW}(1 \colon C^{n-*} \to (SC)^{n+1-*}) = 0.$

7. $\tau^{NEW}((-1)^r \colon C^{n+1-*} \to S(C^{n-*})) = 0.$

8. $\tau^{NEW}((-1)^{(n+1)r} \colon (C^{n-*})^{n-*} \to C) = \frac{n}{2}(n+1)\epsilon(\chi(C), \chi(C))^*$

Proof. Parts 1 to 4 follow straight from the definitions. For part 5:

$$\begin{aligned} &\tau^{NEW}((C \oplus D)^{n-*} \to C^{n-*} \oplus D^{n-*}) \\ &= \eta_{C^{n-*} \oplus D^{n-*}} - \eta_{(C \oplus D)^{n-*}} \\ &= \epsilon(\chi(C^{n-*}), (D^{n-*})_{even}) + (-)^n \epsilon(\chi(C), D_{even})^* \end{aligned}$$

The result follows after considering the odd and even cases.

Part 6 follows straight from the definitions. For part 7:

$$\begin{aligned} &\tau^{NEW}((-1)^r \colon C^{n+1-*} \to S(C^{n-*})) \\ &\quad \tau^{NEW}((-1)^r \colon C^{n+1-*} \to S(C^{n-*})) \\ &= \eta_{S(C^{n-*})} - \eta_{C^{n+1-*}} + \textstyle\sum_{r \equiv n \pmod 2} \epsilon(C_r, C_r)^* \\ &= \alpha_{n+1}(C) + \alpha_n(C) + \textstyle\sum_{r \equiv n \pmod 2} \epsilon(C_r, C_r)^* \\ &= 0 \end{aligned}$$

For part 8:

$$\begin{aligned} &\tau^{NEW}((-1)^{(n+1)r} \colon (C^{n-*})^{n-*} \to C) \\ &= \eta_C - \eta_{(C^{n-*})^{n-*}} + \tau((-)^{(n+1)r} : C_r \to C_r) \\ &= \alpha_n(C^{n-*}) + (-)^n \alpha_n(C)^* + (n+1) \textstyle\sum_{r \text{ odd}} \epsilon(C_r, C_r) \\ &= \textstyle\sum_{r \equiv n+2, n+3 (\mathrm{mod}\ 4)} (\epsilon(C_r, C_r) + \epsilon(C_{n-r}, C_{n-r})) \\ &\quad + (n+1) \textstyle\sum_{r \text{ odd}} \epsilon(C_r, C_r) \\ &= \begin{cases} 0 \\ \epsilon(\chi(C), \chi(C)) \\ \epsilon(\chi(C), \chi(C)) \\ 0 \end{cases} \text{for } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \\ &= \tfrac{n}{2}(n+1)\epsilon(\chi(C), \chi(C)) \end{aligned}$$

□

Lemma 5.17. *The torsion of a contractible signed chain complex C in $\mathbb{A}$ satisfies:*

$$\tau^{NEW}(C^{n-*}) = (-)^{n+1}\tau^{NEW}(C)^* \in K_1^{iso}(\mathbb{A})$$

Proof. We denote by $\bar{C}^{n-*}$ the chain complex with $(\bar{C}^{n-*})_r = (C_{n-r})^*$ and

$$d_{\bar{C}^{n-*}} = d_C^* \colon \bar{C}^{n-r} \to \bar{C}^{n-r+1}$$

We have an isomorphism $f\colon C_r^{n-*} \to \bar{C}_r^{n-*}$ given by $f = -1$ if $r \equiv n+2, n+3 \pmod 4$ and $f = 1$ otherwise. By considering the torsion of this isomorphism we have:

$$\tau(C^{n-*}) = \tau(\bar{C}^{n-*}) + \alpha_n(C) \tag{5.3.1}$$

Let n_{even} be the greatest even integer $\leq n$, similarly n_{odd}. For any chain contraction Γ for C we have the following commutative diagram:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
C^{n_{even}} \oplus \ldots \oplus C^0 \arrow[r, "{\begin{pmatrix} d^* & 0 & 0 & \ldots \\ \Gamma^* & d^* & 0 & \ldots \\ 0 & \Gamma^* & d^* & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}}"] \arrow[d, swap, "{\begin{pmatrix} & & 1 \\ & 1 & \\ \iddots & & \\ 1 & & \end{pmatrix}}"] & C^{n_{odd}} \oplus \ldots \oplus C^1 \arrow[d, "{\begin{pmatrix} & & 1 \\ & 1 & \\ \iddots & & \\ 1 & & \end{pmatrix}}"] \\
C^0 \oplus \ldots \oplus C^{n_{even}} \arrow[r, swap, "{\begin{pmatrix} d^* & \Gamma^* & 0 & \ldots \\ 0 & d^* & \Gamma^* & \ldots \\ 0 & 0 & d^* & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}}"] & C^1 \oplus \ldots \oplus C^{n_{odd}}
\end{tikzcd}$$

The torsion of the lower map in this diagram is $\tau(C)^*$; the torsion of the uppermost map is $(-)^{n+1}\tau(\bar{C}^{n-*})$. So, by first considering the torsions of the maps in the above diagram we have:

$$\begin{aligned}
\tau(\bar{C}^{n-*}) &= (-)^{n+1}\tau(C)^* + (-)^{n+1}\Big(\sum_{i>j;\, i,j \text{ even}} \epsilon(C^i, C^j) - \sum_{i>j;\, i,j \text{ odd}} \epsilon(C^i, C^j)\Big) \\
&= (-)^{n+1}\tau(C)^* + (-)^n\beta(C,C)^*
\end{aligned}$$

Hence by equation 5.3.1

$$\tau(C^{n-*}) = (-)^{n+1}\tau(C)^* + (-)^n\beta(C,C)^* + \alpha_n(C)$$

Using the definition of the dual signed chain complex we have:

$$\tau^{NEW}(C^{n-*}) = (-)^{n+1}\tau^{NEW}(C)^*$$

$\square$

Lemma 5.18. *Let C, D be n-dimensional signed chain complexes in $\mathbb{A}$ and $f : C \to D$ a chain equivalence. Then*

$$\tau^{NEW}(f^{n-*} : D^{n-*} \to C^{n-*}) = (-)^n\tau^{NEW}(f)^* \in K_1^{iso}(\mathbb{A})$$

Proof. We have an isomorphism of chain complexes $\theta : \mathcal{C}(f^{n-*}) \to \mathcal{C}(f)^{n+1-*}$ given by:

$$\mathcal{C}(f^{n-*})_r = C^{n-r} \oplus D^{n-r+1} \xrightarrow{\begin{pmatrix} 0 & (-)^{n-r} \\ 1 & 0 \end{pmatrix}} \mathcal{C}(f)_r^{n+1-*} = D^{n-r+1} \oplus C^{n-r}$$

The torsion of the map θ is given by:

$$\begin{aligned}
\tau^{NEW}(\theta) &= \tau^{NEW}((-)^{n-r} : S(D^{n-*} \to D^{n+1-*}) \\
&\quad +\tau^{NEW}(C^{n-*} \to (SC)^{n+1-*}) \\
&\quad +\tau^{NEW}((SC)^{n+1-*} \oplus D^{n+1-*} \to (SC \oplus D)^{n+1-*}) \\
&\quad +(-)^{n+1}\tau^{NEW}(D \oplus SC \to SC \oplus D)^* \\
&= n\epsilon(\chi(D), \chi(D))^* + (n+1)\epsilon(\chi(D), \chi(D))^* \\
&\quad +\epsilon(\chi(D), \chi(D))^* \\
&= 0
\end{aligned}$$

and the result follows since $\tau^{NEW}(\mathcal{C}(f)^{n+1-*}) = (-)^n\tau^{NEW}(f)^*$ $\square$

Lemma 5.19. *Let C, D be n-dimensional signed chain complexes in $\mathbb{A}$ and $f : C^{n-*} \to D$ a chain equivalence. Then*

$$\tau^{NEW}(Tf : D^{n-*} \to C) = (-)^n\tau^{NEW}(f)^* + \frac{n}{2}(n+1)\epsilon(\chi(C), \chi(C)) \in K_1^{iso}(\mathbb{A})$$

Proof. Using lemmas 5.16 and 5.18 we have:

$$\begin{aligned}
\tau^{NEW}(Tf : D^{n-*} \to C) &= \tau^{NEW}(f^* : D^{n-*} \to (C^{n-*})^{n-*}) \\
&\quad +\tau^{NEW}((-1)^{(n+1)r}(C^{n-*})^{n-*} \to C) \\
&= (-)^n\tau^{NEW}(f)^* + \frac{n}{2}(n+1)\epsilon(\chi(C), \chi(C))
\end{aligned}$$

as required. $\square$

Together the above three lemma prove proposition 5.15.

5.4 The signed Poincaré derived category with involution.

In this section we will add an involution to a particular subcategory of the signed derived category. Let $\mathbb{SPD}_n(\mathbb{A}(R))$ denote the category whose object are signed n-dimensional chain complexes C in $\mathbb{A}(R)$ which are isomorphic to their dual complexes C^{n-*} and $\chi(C) = 0$ if$n$ is odd. Then we have an involution

$$* : C \mapsto C^{n-*}$$

$$* : (f : C \to D) \mapsto (f^{n-*} : D^{n-*} \to C^{n-*})$$

with the natural equivalence $e(C)$ given by

$$e(C) = (-)^{(n+1)r} : C \to (C^{n-*})^{n-*}$$

We call this category the *signed Poincaré derived category with n-involution.* Again we write $\mathbb{SPD}_n(\mathbb{A}(r))$ as $\mathbb{SPD}(r)$. In order to show that this is a covariant functor of additive categories we must show that $*(A \oplus B) = *A \oplus *B$. However, the condition that $\chi(C)$ is odd if n is odd implies that the torsion of the rearrangement map $(C \oplus D)^{n-*} \to C^{n-*} \oplus D^{n-*}$ is trivial (see lemma 5.16 part 5) and the functor $*$ is additive. As in the case of the signed derived category we have a map

$$i_* : K_1^{iso}(\mathbb{SPD}_n(\mathbb{A}(R))) \to K_1^{iso}(\mathbb{A}(R)) \ ; \ \tau^{iso}(f) \mapsto \tau^{NEW}(f)$$

The behaviour of i_* under the involution on $\mathbb{SPD}_n(\mathbb{A}(R))$ is given by

$$i_*(f^*) = (-)^n i_*(f)^*$$

5.5 Torsion of Poincaré complexes

We now move on to consider symmetric Poincaré complexes. We will restrict ourselves to considering symmetric Poincaré complexes over a ring R, that is we work over $\mathbb{A} = \mathbb{A}(R)$ and we will consider the torsion invariants to lie in the more familiar $K_1(R)$. We will define the notion of the absolute torsion of a symmetric Poincaré complex to be, essentially, the torsion of the Poincaré duality chain equivalence.

A signed symmetric (Poincaré) complex is a symmetric (Poincaré) complex (C, ϕ_0) where in addition C is a signed chain complex. Such a complex is said to be *round* or *even* if C is round or even respectively.

Lemma 5.20. *The boundary $(\partial C, \partial\phi)$ of any signed n-dimensional symmetric complex (C, ϕ) satisfies*

$$\tau^{NEW}(\partial\phi_0 : (\partial C)^{n-1-*} \to \partial C) = \frac{n}{2}(n+1)\epsilon(\chi(C), \chi(C)) \in K_1(R)$$

Proof. The map

$$\partial\phi_0 = \begin{pmatrix} (-)^{n-r-1}T\phi_1 & (-)^{rn} \\ 1 & 0 \end{pmatrix} : \partial C^{n-r-1} \to \partial C_r$$

is an isomorphism.

We have that

$$\begin{aligned}
\tau^{NEW}(\partial\phi_0) &= \tau^{NEW}\left(\begin{pmatrix} 0 & (-)^r n \\ 1 & 0 \end{pmatrix} : (\Omega C \oplus C^{n-*})^{n-1-*} \to \Omega C \oplus C^{n-*}\right) \\
&= \tau^{NEW}((\Omega C \oplus C^{n-*})^{n-1-*} \to (\Omega C)^{n-1-*} \oplus (C^{n-*})^{n-1-*}) \\
&\quad + \tau^{NEW}((C^{n-*})^{n-1-*} \to (\Omega C^{n-1-*})^{n-1-*}) \\
&\quad + \tau^{NEW}((-)^{nr} : (\Omega C^{n-1-*})^{n-1-*} \to \Omega C) \\
&\quad + \tau^{NEW}((\Omega C)^{n-1-*} \to C^{n-*}) \\
&= \tau^{NEW}(C^{n-*} \oplus \Omega C \to \Omega C \oplus C^{n-*}) \\
&= \frac{n}{2}(n+1)\epsilon(\chi(C), \chi(C))
\end{aligned}$$

using the results of lemma 5.16. □

We can now define a new absolute torsion invariant of Poincaré complexes which is additive and a cobordism invariant.

Definition 5.21. *We define the* absolute torsion *of a signed Poincaré complex (C, ϕ) as*

$$\tau^{NEW}(C, \phi) = \tau^{NEW}(\phi_0) \in K_1(R)$$

Proposition 5.22. *Let (C, ϕ) and (C', ϕ') be signed n-dimensional Poincaré complexes. Then:*

1. Additivity:

$$\tau^{NEW}(C \oplus C', \phi \oplus \phi') = \tau^{NEW}(C, \phi) + \tau^{NEW}(C'\phi'_0) \in K_1(R)$$

2. Duality:

$$\tau^{NEW}(C, \phi) = (-)^n \tau^{NEW}(C, \phi)^* + \frac{n}{2}(n+1)\epsilon(\chi(C), \chi(C)) \in K_1(R)$$

(n.b. the above sign term disappears in the case where $\mathbb{A} = \mathbb{A}(R)$ and where anti-symmetric form over the ring R necessarily have even rank; this is the case for $R = \mathbf{Z}$ or $R = \mathbf{Q}$ but not $R = \mathbf{C}$).

3. Homotopy invariance: *Suppose (f, σ_s) is a homotopy equivalence from (C, ϕ) to (C', ϕ'). Then*

$$\tau^{NEW}(C', \phi') = \tau^{NEW}(C, \phi) + \tau(f) + (-)^n \tau(f)^* \in K_1(R)$$

4. Cobordism Invariance: *Suppose that (C, ϕ) is homotopy equivalent to the boundary of some $(n+1)$-dimensional symmetric complex with torsion (D, ϕ^D). Then*

$$\begin{aligned}\tau^{NEW}(C, \phi) &= (-)^{n+1} \tau^{NEW}(C \to \partial D)^* - \tau^{NEW}(C \to \partial D) \\ &\quad + \frac{1}{2}(n+1)(n+2)\epsilon(\chi(D), \chi(D)) \in K_1(R)\end{aligned}$$

5. Orientation change:

$$\tau^{NEW}(C, -\phi) = \tau^{NEW}(C, \phi) + \epsilon(\chi(C), \chi(C)) \in K_1(R)$$

6. The absolute torsion of a signed Poincaré complex is independent of the choice of sign η_C.

Proof. 1. A symmetric Poincaré complex of odd dimension satisfies $\chi(C) = 0$, hence the map $(C \oplus C')^{n-*} \to C^{n-*} \oplus C'^{n-*}$ has trivial absolute torsion. Additivity now follows from the additivity of chain equivalences.

2. We know that ϕ_0 is homotopic to $T\phi_0$; duality now follows by applying lemma 5.19.

3. We have that $\phi_0' \simeq f\phi_0 f^*$ and hence

$$\tau^{NEW}(\phi_0') = \tau^{NEW}(f) + \tau^{NEW}(\phi_0) + (-)^n \tau^{NEW}(f)^*$$

4. This follows from proposition 5.20 and homotopy invariance.

5. We have that $\tau^{NEW}(-\phi_0) = \tau^{NEW}(\phi_0) + \tau^{NEW}(-1 : C \to C) = \tau^{NEW}(\phi_0) + \chi(C)\tau(-1)$.

6. A change in η_C leads to a corresponding change in $\eta_{C^{n-*}}$ so $\tau^{NEW}(\phi_0)$ is unchanged.

□

5.6 Round L-theory

We refer the reader to [HRT87] for the definition of the round symmetric L-groups $L^n_r(A)$. The absolute torsion defined in this paper as $\tau(C, \phi_0) = \tau(\phi_0)$ (here $\tau(\phi_0)$ refers to the absolute torsion defined in [Ran85]) is not a cobordism invariant. We can define such an invariant using the absolute torsion of a Poincaré complex. If this invariant is substituted for $\tau(C, \phi_0)$ as defined in [HRT87] then the results become correct.

Lemma 5.23. *Let (C, ϕ) be a round Poincaré complex. The reduced element*

$$\tau^{NEW}(C, \phi) \in \widehat{H}^n(\mathbf{Z}_2; K_1(R))$$

is independent of the choice of sign η_C; moreover we have a well defined homomorphism:

$$L^n_r(R) \to \widehat{H}^n(\mathbf{Z}_2; K_1(R))$$

given by $(C, \phi) \mapsto \tau^{NEW}(C, \phi)$.

Proof. The element $\tau^{NEW}(C, \phi) \in \widehat{H}^n(\mathbf{Z}_2; K_1(R))$ is independent of the choice of sign by proposition 5.22 part 6. The absolute torsion is additive by proposition 5.22 part 1. The absolute torsion of the boundary of a round symmetric complex is trivial in the reduced group $\widehat{H}^n(\mathbf{Z}_1; K_1(R))$ by proposition 5.22 part 4. Hence the torsion of a round null-cobordant complex is trivial and the map

$$L^n_r(R) \to \widehat{H}^n(\mathbf{Z}_2; K_1(R))$$

given by $(C, \phi) \mapsto \tau^{NEW}(C, \phi)$ is well defined. □

5.7 Applications to Poincaré spaces.

5.7.1 The absolute torsion of Poincaré spaces.

To any n-dimensional Poincaré space X we may associate via the symmetric construction a symmetric Poincaré complex $(C(\widetilde{X}), \phi)$ over the ring $R = \mathbf{Z}[\pi_1(X)]$, well defined up to homotopy equivalence. By property 2 of proposition 5.22 the absolute torsion of such a Poincaré complex satisfies

$$\tau^{NEW}(C, \phi) = (-)^n \tau^{NEW}(C, \phi)^*$$

since $\chi(X) \equiv 0 \pmod 2$ unless $n \equiv 0 \pmod 4$. Hence the torsion $\tau^{NEW}(C, \phi)$ may be considered to lie in the group $\widehat{H}^n(\mathbf{Z}_2; K_1(\mathbf{Z}[\pi_1(X)]))$. By property 3 of proposition 5.22 if (C, ϕ) is homotopy equivalent to (C', ϕ') then

$$\tau^{NEW}(C, \phi) = \tau^{NEW}(C', \phi') \in \widehat{H}^n(\mathbf{Z}_2; K_1(\mathbf{Z}[\pi_1(X)]))$$

hence

$$\tau^{NEW}(X) := \tau^{NEW}(C,\phi) \in \widehat{H}^n(\mathbf{Z}_2; \mathbf{Z}[\pi_1(X)])$$

is well defined.

5.7.2 Examples of the absolute torsion of Poincaré spaces.

5.7.2.1 The circle

We may associate to the circle (S^1) the following chain complex over $R = \mathbf{Z}[\pi_1(S^1)] = \mathbf{Z}[t,t^{-1}]$ by giving it the CW-decomposition consisting of one 1-cell and one 0-cell:

$$\begin{tikzcd} \mathbf{Z}[t,t^{-1}] \arrow[r, "1"] \arrow[d, "t^{-1}-1"'] & \mathbf{Z}[t,t^{-1}] \arrow[d, "1-t"] \\ \mathbf{Z}[t,t^{-1}] \arrow[r, "t"] & \mathbf{Z}[t,t^{-1}] \end{tikzcd}$$

In this diagram the two modules on the right are the chain complex, the two modules on the left are the dual complex and the sideways arrows represent ϕ_0.

Hence $\tau^{NEW}(S^1) = \tau(-t) \in \widehat{H}^n(\mathbf{Z}_2; K_1(\mathbf{Z}[t,t^{-1}])$.

5.7.2.2 The absolute torsion of an algebraic mapping torus.

The mapping torus of a map $f : X \to X$ is the space obtained from $X \times I$ obtained by attaching the boundaries $X \times \{0\}$ and $X \times \{1\}$ using the map f. The following algebraic analogue is defined by Ranicki ([Ran98] definition 24.3, the reader should note the different sign convention used here).

Definition 5.24. *The* algebraic mapping torus *of a morphism $(f,\sigma) : (C,\phi) \to (C,\phi)$ from an n-dimensional symmetric Poincaré complex (C,ϕ) over a ring R to itself is the $(n+1)$-dimensional symmetric complex $(T(f),\theta)$ over the ring $R[z,z^{-1}]$ defined by:*

$$T(f) = \mathcal{C}(f-z)$$

$$\theta_0 = \begin{pmatrix} (-)^n\sigma_0 & \phi_0 z \\ (-)^{n-r+1}\phi_0 f^* & 0 \end{pmatrix} : T(f)^{n-r+1} \to T(f)_r$$

The complex is Poincaré if the morphism f is a chain equivalence.

Lemma 5.25. *Let $(f,\sigma) : (C,\phi) \to (C,\phi)$ be a self chain equivalence from an n-dimensional symmetric Poincaré complex (C,ϕ) over a ring R to itself. Then:*

$$\tau^{NEW}(T(f),\theta) = \tau^{NEW}(f) + \tau^{NEW}(-z : C \to C) \in K_1(R[z,z^{-1}])$$

Proof. We have a commutative diagram with short exact rows:

$$\begin{tikzcd}[column sep=large]
0 \arrow[r] & C^{n-r} \arrow[r, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] \arrow[d, "-z\phi_0"'] & \mathcal{C}(f-z)^{n+1-r} \arrow[r, "\begin{pmatrix} (-)^r & 0 \end{pmatrix}"] \arrow[d, "\theta"] & S(C^{n-r}) \arrow[r] \arrow[d, "(-)^n\phi_0 f^*"] & 0 \\
0 \arrow[r] & C_r \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"'] & \mathcal{C}(f-z)_r \arrow[r, "\begin{pmatrix} 0 & 1 \end{pmatrix}"'] & SC_r \arrow[r] & 0
\end{tikzcd}$$

The absolute torsion of the lower short exact sequence is trivial; for the top map we have:

$$\begin{aligned}
&\tau^{NEW}\left(C^{n-r}, \mathcal{C}(f-z)^{n+1-r}, S(C^{n-r}); \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} (-)^r & 0 \end{pmatrix}\right) = \\
&\tau^{NEW}\left(\begin{pmatrix} 0 & (-)^r \\ 1 & 0 \end{pmatrix} : (C \oplus SC)^{n+1-*} \to C^{n-*} \oplus S(C^{n-*})\right) \\
&= \tau^{NEW}((C \oplus SC)^{n+1-*} \to C^{n+1-*} \oplus (SC)^{n+1-*}) \\
&\quad +\tau^{NEW}((-1)^r : C^{n+1-*} \to S(C^{n-*})) \\
&\quad +\tau^{NEW}((SC)^{n+1-*} \to C^{n-*}) \\
&\quad +\tau^{NEW}(S(C^{n-*}) \oplus C^{n-*} \to C^{n-*} \oplus S(C^{n-*})) \\
&= 0
\end{aligned}$$

Using proposition 5.12 part 4 we have:

$$\begin{aligned}
\tau^{NEW}(T(f), \theta) &= \tau^{NEW}(\theta) \\
&= \tau^{NEW}(-z\phi_0) + \tau^{NEW}((-1)^n S(\phi_0 f^*) : S(C^{n-*}) \to SC) \\
&= -\tau^{NEW}(f^*) + \tau^{NEW}(-z : C \to C) \\
&= (-)^{n+1-*}\tau^{NEW}(f)^* + \tau^{NEW}(-z : C \to C) \\
&= \tau^{NEW}(f) + \tau^{NEW}(-z : C \to C)
\end{aligned}$$

as required. □

5.7.2.3 A specific example of a mapping torus.

We return to the example of the orientation preserving self-homeomorphism $f : \mathbf{CP}^2 \to \mathbf{CP}^2$ given by complex conjugation in some choice of homogeneous coordinates (see section 5.1.6). We recall that the torsion of this map is $\tau^{NEW}(f) = \tau(-1) \in K_1(\mathbf{Z})$. Using lemma 5.25 we compute the absolute torsion of the mapping torus of f as

$$\tau^{NEW}(T(f)) = \tau(z^3) \in K_1(\mathbf{Z}[z, z^{-1}])$$

where z is a generator of $\pi_1(T(f)) = \pi_1(S^1) = \mathbf{Z}$. By contrast we may compute the absolute torsion of the space $T(Id : \mathbf{CP}^2 \to \mathbf{CP}^2) = S^1 \times \mathbf{CP}^2$ as

$$\tau^{NEW}(S^1 \times \mathbf{CP}^2) = \tau(-z^3) \in K_1(\mathbf{Z}[z, z^{-1}])$$

hence the absolute torsion can distinguish between these two $\mathbf{CP}^2$ bundles over S^1.

5.8 Identifying the sign term

Throughout this section we work over a group ring $R = \mathbf{Z}[\pi]$ for some group π (or, more generally, any ring with involution R which admits a map $R \to \mathbf{Z}$ such that the composition $\mathbf{Z} \to R \to \mathbf{Z}$ is the identity). We first identify the relationship between the "sign" term of the absolute torsion of a Poincaré complex and the traditional signature and Euler characteristic and semi-characteristic invariants.

We have a canonical decomposition of $K_1(\mathbf{Z}[\pi])$ as follows:

$$K_1(\mathbf{Z}[\pi]) = \widetilde{K_1}(\mathbf{Z}[\pi]) \oplus \mathbf{Z}_2$$

with the $\mathbf{Z}_2$ component the "sign" term identified by the map

$$i_* K_1(\mathbf{Z}[\pi]) \to K_1(\mathbf{Z}) = \mathbf{Z}_2$$

induced by the augmentation map $i : \pi \to 1$ (more generally, a map $R \to \mathbf{Z}$ gives a map $K_1(R) \to K_1(\mathbf{Z}) = \mathbf{Z}_2$ which gives a splitting $K_1(R) = \widetilde{K_1}(R) \oplus \mathbf{Z}_2$). We wish to determine the $\mathbf{Z}_2$ component in terms of more traditional invariants of Poincaré complexes. The augmentation map may also be applied to a symmetric complex (C, ϕ) over $\mathbf{Z}[\pi]$ to form a symmetric complex over $\mathbf{Z}$ by forgetting the group. Functoriality of the absolute torsion tells us that this complex has the same sign term as (C, ϕ), hence to identify the sign term it is sufficient to consider symmetric Poincaré complexes over $\mathbf{Z}$. We will require the Euler semi-characteristic $\chi_{1/2}(C)$ of Kervaire [Ker56]

Definition 5.26. *The* Euler semi-characteristic $\chi_{1/2}(C)$ *of a* $(2k-1)$*-dimensional chain complex* C *over a field* F *is defined by*

$$\chi_{1/2}(C) = \sum_{i=0}^{k-1} (-)^i \operatorname{rank}_F H_i(C) \in \mathbf{Z}$$

For a $(2k-1)$*-dimensional chain complex* C *over* $\mathbf{Z}$ *we define*

$$\chi_{1/2}(C; F) = \chi_{1/2}(C \otimes_{\mathbf{Z}} F)$$

Proposition 5.27. *The absolute torsion of an* n*-dimensional symmetric Poincaré complex over* $\mathbf{Z}$ *is determined by the signature and the Euler characteristic and semi-characteristic as follows:*

1. *If $n = 4k$ then:*

$$\tau^{NEW}(C,\phi) = \frac{\mathrm{sign}(C,\phi) - (1+2k)\chi(C)}{2}\tau(-1)$$

with $\sigma(C)$ the signature of the complex.

2. *If $n = 4k+1$ then $\tau^{NEW}(C,\phi) = \chi_{1/2}(C;\mathbf{Q})$.*

3. *Otherwise $\tau^{NEW}(C,\phi) = 0$.*

As an example we have a simple corollary:

Corollary 5.28. *The absolute torsion of a self homotopy equivalence of a $4k+2$-dimensional symmetric Poincaré complex over $\mathbf{Z}$ is trivial.*

Proof. Let (C,ϕ) be the symmetric Poincaré complex and $f : C \to C$ the self-homotopy equvialence. Then the algebraic mapping torus $T(f)$ is a $4k+3$-dimensional symmetric Poincaré complex and by lemma 5.25 we have

$$\tau^{NEW}(T(f)) = \tau^{NEW}(f) \in K_1(\mathbf{Z}[z,z^{-1}])$$

The augmentation map $\epsilon : \mathbf{Z} \to 1$ induces a map of rings $\epsilon_* : \mathbf{Z}[z,z^{-1}] \to \mathbf{Z}$. Since L-theory and the absolute torsion are functorial, $\epsilon_* T(f)$ represents an element of $L^{4k+3}(\mathbf{Z})$ with absolute torsion $\tau^{NEW}(\epsilon_* T(f)) = \tau^{NEW}(f) \in K_1(\mathbf{Z})$. However by part 3 of the above proposition $\tau^{NEW}(\epsilon_* T(f)) = 0$. □

The aim of the rest of this section is to prove proposition 5.27 . We recall from [Ran80b] the computation of the symmetric L-groups $L^n_h(\mathbf{Z})$ of the integers $\mathbf{Z}$:

$$L^n_h(\mathbf{Z}) = \begin{cases} \mathbf{Z} \text{ (signature)} \\ \mathbf{Z}_2 \text{ (de Rham invariant)} \\ 0 \\ 0 \end{cases} \text{for } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \pmod 4$$

The deRham invariant $d(C) \in \mathbf{Z}_2$ of a $(4k+1)$-dimensional Poincaré complex was expressed in [LMP69] as the difference

$$d(C) = \chi_{1/2}(C;\mathbf{Z}_2) - \chi_{1/2}(C;\mathbf{Q})$$

For dimensions $n \equiv 2,3 \pmod 4$ the absolute torsion is a cobordism invariant (proposition 5.22 part 4) so the above computation of the symmetric L-groups tells us that the absolute torsion is trivial in these cases, thus proving the third part of proposition 5.27.

If $n = 4k+1$ then the absolute torsion is not a cobordism invariant; however it is a *round* cobordism invariant, so absolute torsion defines a map:

$$L_{rh}^{4k+1}(\mathbf{Z}) \to K_1(\mathbf{Z})$$

Since $\chi(C) = 0$ for all odd-dimensional symmetric Poincaré complexes every such $(4k+1)$-dimensional complex represents an element in $L_{rh}^{4k+1}(\mathbf{Z})$. In [HRT87] (proposition 4.2) the group $L_{rh}^{4k+1}(\mathbf{Z})$ is identified as:

$$L_{rh}^{4k+1}(\mathbf{Z}) = \mathbf{Z}_2 \oplus \mathbf{Z}_2; \quad C \mapsto (\chi_{1/2}(C;\mathbf{Z}_2), \chi_{1/2}(C,\mathbf{Q}))$$

We now construct explicit generators of this group and compute their absolute torsions. We define the generator (G,ϕ) to have chain complex G concentrated in dimensions $2k$ and $2k+1$ defined by:

$$d_G = 0 : G_{2k+1} = \mathbf{Z} \to G_{2k} = \mathbf{Z}$$

with the morphisms ϕ given by:

$$\phi_0 = \begin{cases} 1 : G^{2k} = \mathbf{Z} \to G_{2k+1} = \mathbf{Z} \\ 1 : G^{2k+1} = \mathbf{Z} \to G_{2k} = \mathbf{Z} \end{cases} \qquad \phi_1 = 0$$

Geometrically (G,ϕ) is the symmetric Poincaré complex over $\mathbf{Z}$ associated to the circle. By direct computation, $\chi_{1/2}(G,\mathbf{Z}_2) = 1$, $\chi_{1/2}(G,\mathbf{Q}) = 1$ and $\tau^{NEW}(G,\phi) = \tau(-1)$. We define the generator (H,ψ) to have chain complex H concentrated in dimensions $2k$ and $2k+1$ defined by:

$$d_H = 2 : H_{2k+1} = \mathbf{Z} \to H_{2k} = \mathbf{Z}$$

with the morphisms ψ given by:

$$\psi_0 = \begin{cases} -1 : H^{2k} = \mathbf{Z} \to H_{2k+1} = \mathbf{Z} \\ 1 : H^{2k+1} = \mathbf{Z} \to H_{2k} = \mathbf{Z} \end{cases} \qquad \psi_1 = 1 : H^{2k+1} \to H_{2k+1}$$

Geometrically (H,ψ) is a symmetric Poincaré complex over $\mathbf{Z}$ which is cobordant to the complex associated to the mapping torus of the self-diffeomorphism of $\mathbf{CP}^2$ given by complex conjugation. Again by direct computation, $\chi_{1/2}(H,\mathbf{Z}_2) = 1$, $\chi_{1/2}(H,\mathbf{Q}) = 0$ and $\tau^{NEW}(H,\psi) = 0$. By considering the absolute torsion of these two generators we see that the map $L_{rh}^{4k+1}(\mathbf{Z}) \to K_1(\mathbf{Z})$ is given by:

$$(C,\phi) \mapsto \chi_{1/2}(C,\mathbf{Q})\tau(-1)$$

thus proving part two of proposition 5.27.

To prove part 1 of proposition 5.27 we use the following lemma (from [HKR05]):

Lemma 5.29. *We have the following relationship between τ^{NEW} and signature modulo 4 of a $4k$-dimensional Poincaré complex (C, ϕ):*

$$\mathrm{sign}(C, \phi) = 2\tau^{NEW}(C, \phi) + (2k+1)\chi(C) \in \mathbf{Z}_4$$

where the map $2 : K_1(\mathbf{Z}) = \mathbf{Z}_2 \to \mathbf{Z}_4$ takes $\tau(-1)$ to $2 \in \mathbf{Z}_4$.

Proof. The right-hand side is clearly additive. Suppose momentarily that C is null-cobordant, in other words that C is homotopy equivalent to the boundary of some $(n+1)$-dimensional symmetric complex (D, ϕ^D). Then by proposition 5.22 part 4:

$$\tau^{NEW}(C) = \epsilon(\chi(D), \chi(D))$$

and one can easily see that $\chi(D) = \frac{1}{2}\chi(C)$ so

$$2\tau^{NEW}(C, \phi) + (2k+1)\chi(C) = 0 \in \mathbf{Z}_4$$

in this case. Hence the right-hand side is a cobordism invariant so we only have to show that the formula holds for a generator of $L^{4k}(\mathbf{Z}) \cong \mathbf{Z}$. One can easily check this. $\square$

A simple rearrangement of the formula of the above lemma yields the first part of proposition 5.27.

Chapter 6

The absolute torsion of a fibre bundle.

For the sake of completeness we now prove a formula for the absolute torsion of the total space of a Poincaré fibration satisfying assumption 1.2. This first appeared in [HKR05] in the special case of a fibre bundle of PL-manifolds. We will use the results of [HKR05] concerning the absolute torsion of filtered complexes. Unlike in [HKR05] we will not use this result to prove that the signature of a fibration is multiplicative modulo four, instead we will compute the absolute torsion of a twisted product of a symmetric Poincaré complex and a $(\mathbf{Z}, m)$-symmetric representation.

We first recall from [Lüc86] and [LR88] the construction of the algebraic K-theory transfer map associated to a fibration. Let $p : E \to B$ be a fibration satisfying assumption 1.2. Then we have an associated transfer functor

$$- \otimes (C(\hat{F}), U) : \mathbb{A}^{based}(\mathbf{Z}[\pi_1(B)]) \to \mathbb{D}(\mathbf{Z}[\pi_1(E)])$$

If we choose some sign (arbitrarily) for $C(\hat{F})$ we can assume that this functor lands in the signed derived category $\mathbb{SD}(\mathbf{Z}[\pi_1(E)])$, since the functor applied objects $\mathbf{Z}[\pi_1(B)]^k$ is $\bigoplus_k C(\hat{F})$ a well-defined signed complex. If we take K_1^{iso} of this functor we have a homomorphism:

$$K_1(\mathbf{Z}[\pi_1(B)]) \to K_1^{iso}(\mathbb{SD}(\mathbf{Z}[\pi_1(E)]))$$

If we now compose with the forgetful map $i_* : K_1^{iso}(\mathbb{SD}(\mathbf{Z}[\pi_1(E)])) \to K_1(\mathbf{Z}[\pi_1(E)])$ of proposition 5.14 we have a map:

$$p^! : K_1(\mathbf{Z}[\pi_1(B)]) \to K_1(\mathbf{Z}[\pi_1(E)])$$

Note that this depends only on the action of $\pi_1(B)$ on the fibre (in particular it doesn't depend on the choice of sign for $C(\hat{F})$).

We can now state the main Theorem of this chapter:

Theorem 6.1. *Let $p : F^m \to E^{m+n} \to B^n$ be a Poincaré fibration satisfying assumption 1.2. Then:*

$$\tau^{NEW}(E) = p^!\tau^{NEW}(B) + \chi(B)q_*\tau^{NEW}(F) \in \hat{H}^{n+m}(\mathbf{Z}_2; K_1(\mathbf{Z}[\pi_1(E)]))$$

where $q_* : K_1(\mathbf{Z}[\pi_1(F)]) \to K_1(\mathbf{Z}[\pi_1(E)])$ is the map induced by the inclusion $F \to E$.

6.1 The absolute torsion of filtered chain complexes.

In this chapter we recall the definitions and results (without proof) of [HKR05] chapter 12. First we must define signed analogues of filtered complexes and associated complexes (definitions 3.1 and 3.3).

Definition 6.2.

- *A* signed k-filtered complex (F_*C, η_{G_*C}) *in $\mathbb{A}$ consists of a filtered complex F_*C together with signs:*

$$\eta_{G_*C} \in K_1^{iso}(\mathbb{SD}(\mathbb{A})) \qquad (6.1.1)$$
$$\eta_{G_j} \in K_1^{iso}(\mathbb{A}) \qquad (6.1.2)$$

  for all $0 \le j \le k$. A map of such complexes is a map of the underlying filtered complexes.

- *Given a signed filtered complex (F_*C, η_{G_*C}) we define the* signed associated complex (G_*C, η_{G_*C}) *to be the signed chain complex in $\mathbb{SD}(\mathbb{A})$ whose underlying unsigned chain complex in $\mathbb{D}(\mathbb{A})$ is the usual associated complex, but with signed objects (G_jC, η_{G_jC}) and "global" sign η_{G_*C}.*

- *Given a signed filtered complex (F_*C, η_{G_*C}) we define the unfiltered signed complex (C, η_C) by:*

$$\eta_C = i_*\eta_{G_*C} + \eta_{G_0C \oplus S(G_1C \oplus S(G_2C \oplus \ldots \oplus SG_kC)\ldots)}$$

The best justification for the sign terms appearing in this definition is the following theorem ([HKR05] Theorem 12.17):

Theorem 6.3. *Let $F_*f : (F_*C, \eta_{G_*C}) \to (F_*D, \eta_{G_*D})$ be a filtered homotopy equivalence of signed filtered complexes. Then the associated map $G_*f : (G_*C, \eta_{G_*C}) \to (G_*D, \eta_{G_*D})$ is a chain equivalence of signed complexes in $\mathbb{SD}(\mathbb{A})$; moreover*

$$\tau^{NEW}(f : (C, \eta_C) \to (D, \eta_D)) = i_*\tau^{NEW}(G_*f) \in K_1^{iso}(\mathbb{A})$$

We now introduce sign terms for the filtered dual complex (definition 4.2). Let $\mathbb{A}$ be an additive category with involution

Definition 6.4. *1. A filtered complex F_*C in an additive category with involution $\mathbb{A}$ is said to be m-*admissible *if each G_rC is isomorphic to its m-dual $(G_rC)^{m-*}$. In addition, if m is odd we require that $\chi(G_r) = 0$. These conditions are equivalent to saying that G_*C lifts to the category $\mathbb{SPD}_m(\mathbb{A})$.*

2. The signed (n, m)-filtered dual $(F_*^{n,m}C, \eta_{G_*(F_*^{n,m}C)})$ *complex of an m-admissible signed filtered complex (F_*C, η_{F_*C}) to be the signed filtered complex who's underlying filtered chain complex is the (n, m)-filtered dual of F_*C and with filtered signs given by:*

$$\eta_{G_*(F_*^{n,m}C)} = \eta_{(G_*C)^{n-*}} \in K_1^{iso}(\mathbb{SD}(\mathbb{A}))$$

$$\eta_{G_j(F_*^{n,m}C)} = \eta_{(G_jC)^{m-*}} \in K_1^{iso}(\mathbb{A})$$

Observe that the associated complex of the signed filtered dual is the dual of the filtered complex; in other words

$$G_*(F_*^{n,m}C, \eta_{G_*(F_*^{n,m}C)}) = ((G_*C)^{n-*}, \eta_{(G_*C)^{n-*}})$$

as signed chain complexes in $\mathbb{SPD}_m(\mathbb{A})$. We recall Proposition 12.26 of [HKR05]

Proposition 6.5. *Let F_*C be a m-admissible signed filtered chain complex over $\mathbb{A}(R)$. Then*

$$\tau^{NEW}(\theta_{F_*C} : F^{n,m}C \to C^{n+m-*}) = 0$$

as a map between unfiltered signed complexes.

6.2 Completing the proof

The proof of Theorem 6.1. Let $(C(\widetilde{B}), \phi^B)$ be a symmetric complex representing B. By theorem 4.5 there exists a symmetric complex $(C(\widetilde{E}), \phi^E)$ representing E with $C(\widetilde{E})$ a filtered complex and an isomorphism $\mathcal{E} : G_*C(\widetilde{E}) \to C(\widetilde{B}) \otimes (C(\hat{F}), \phi_0^F, U)$ such that $\phi_0^E \circ \theta_{F_*C(\widetilde{E})}$ is a filtered map and

$$\mathcal{E} \circ G_*(\phi_0^E \circ \theta_{F_*C(\widetilde{E})}) \circ \mathcal{E}^{n-*} = (\phi_0^B \otimes (C(\hat{F}), \phi_0^F, U)) \circ H(C(\widetilde{B}))$$

The existence of $\mathcal{E}$ and the fact that $C(\hat{F})$ is isomorphic to $C(\hat{F})^{m-*}$ imply that $C(\widetilde{E})$ is m-admissible. We choose signs to make $C(\widetilde{E})$ a filtered signed complex and $C(\widetilde{B})$ a signed complex. Of course the absolute torsion computed will be independent of these choices. Applying Proposition 6.5 and Theorem 6.3 we see that

$$\begin{aligned}
\tau^{NEW}(E) &= \tau^{NEW}(\phi_0^E) \\
&= \tau^{NEW}(\phi_0^E \circ \theta_{F_*C(\widetilde{E})}) \\
&= i_*\tau^{NEW}(G_*(\phi_0^E \circ \theta_{F_*C(\widetilde{E})})) \\
&= i_*\tau^{NEW}(\mathcal{E}^{-1} \circ (\phi_0^B \otimes (C(\hat{F}), \phi_0^F, U)) \circ H(C(\widetilde{B})) \circ (\mathcal{E}^{-1})^{n-*}) \\
&= i_*\tau^{NEW}(\mathcal{E}^{-1}) + i_*((-)^m \tau^{NEW}(\mathcal{E}^{-1})^*) \\
&\quad + i_*\tau^{NEW}(H(C(\widetilde{B}))) + i_*\tau^{NEW}(\phi_0^B \otimes (C(\hat{F}), \phi_0^F, U)) \\
&= i_*\tau^{NEW}(\mathcal{E}) + (-)^{n+m}(i_*\tau^{NEW}(\mathcal{E}))^* \\
&\quad + i_*\tau^{NEW}(H(C(\widetilde{B}))) + i_*\tau^{NEW}(\phi_0^B \otimes (C(\hat{F}), \phi_0^F, U)) \\
&= i_*\tau^{NEW}(H(C(\widetilde{B}))) + i_*\tau^{NEW}(\phi_0^B \otimes (C(\hat{F}), \phi_0^F, U)) \\
&\qquad \in \hat{H}^{n+m}(\mathbf{Z}_2; K_1(\mathbf{Z}[\pi_1(E)])) \qquad (6.2.1)
\end{aligned}$$

By the functoriality of absolute torsion and the definition of the transfer map

$$i_*\tau^{NEW}(\phi_0^B \otimes (C(\hat{F}), \phi_0^F, U)) = i_*(\tau^{NEW}(\phi_0^B) \otimes (C(\hat{F}), \phi_0^F, U)) = p^!\tau^{NEW}(\phi_0^B) \qquad (6.2.2)$$

By the definition of the absolute torsion of a chain isomorphism:

$$\begin{aligned}
i_*\tau^{NEW}(H(C(\widetilde{B}))) &= \sum_{r=0}^{\infty}(-)^r i_*\tau^{iso}(H(C(\widetilde{B})_r)) \\
&= \sum_{r=0}^{\infty}(-)^r \tau^{NEW}(H(C(\widetilde{B})_r)) \\
&= \sum_{r=0}^{\infty}(-)^r \tau^{NEW}(\bigoplus_{\mathrm{rank}(C(\widetilde{B})_r)} \phi_0^F) \\
&= \chi(B).\tau^{NEW}(\phi_0^F) \qquad (6.2.3)
\end{aligned}$$

If we now substitute the expressions of 6.2.2 and 6.2.3 into equation 6.2.1 we obtain:

$$\tau^{NEW}(E) = p^!\tau^{NEW}(B) + \chi(B)q_*\tau^{NEW}(F) \in \hat{H}^{n+m}(\mathbf{Z}_2; K_1(\mathbf{Z}[\pi_1(E)]))$$

as required. □

Chapter 7

The signature of a fibration modulo four.

In this chapter we will extend the result of [HKR05] that the signature of a PL-fibre bundle of manifolds is multiplicative modulo four to the case of a fibration of Poincaré spaces with the base space satisfying a Whitehead torsion condition (Theorem 7.2). The main step is to prove the corresponding algebraic Theorem for $(\mathbf{Z}, m)$-symmetric representations and visible symmetric Poincaré complexes:

Theorem 7.1. *Suppose that (C, ϕ) is an n-dimensional visible symmetric Poincaré complex over $\mathbb{A}(\mathbf{Z}[\pi])$ such that $\tau(C, \phi) = 0 \in \mathrm{Wh}(\mathbf{Z}[\pi])$ and (A, α, U) a $(\mathbf{Z}, m)$-symmetric representation of $\mathbf{Z}[\pi]$. Then*

$$\mathrm{sign}((C, \phi) \otimes (A, \alpha, U)) = \mathrm{sign}(A, \alpha, U)\mathrm{sign}(C, \phi) \pmod 4$$

The main Theorem follows straight from this:

Theorem 7.2. *Let $F^m \to E^{n+m} \to B^n$ be a Poincaré fibration satisfying assumption 1.2 such that $\tau(B) = 0 \in \mathrm{Wh}(\pi_1(B))$. Then*

$$\mathrm{sign}(E) = \mathrm{sign}(B)\mathrm{sign}(F) \pmod 4$$

Proof. We know that the theorem is true if the dimension of F is odd so we assume that the dimension of F is $2m$. From theorem 4.9 the signature of E is the signature of the twisted tensor product of the symmetric complex $(C(\widetilde{B}), \phi^B)$ with the $\mathbf{Z}$-coefficient bundle $(H_m(F)/torsion, \phi_0^F, U)$ of $\pi_1(B)$ (Theorem 4.9). However the above Theorem show that the signature of such a twisted tensor product is multiplicative modulo four. □

It remains to prove Theorem 7.1. We will describe the absolute torsion of the twisted tensor product of a symmetric complex and a $(\mathbf{Z}, m)$-symmetric representation. Both the statement and proof are very similar to Theorem 6.1 but the advantage here is that the technology of the torsion of filtered complexes is not required.

Definition 7.3. *We define the* transfer map *associated to a* $(\mathbf{Z}, m)$*-symmetric representation* (A, α, U) *to be the map*

$$U^! : K_1(\mathbf{Z}[\pi]) \to K_1(\mathbf{Z})$$

induced by the functor $- \otimes (A, \alpha, U) : \mathbb{A}(\mathbf{Z}[\pi]) \to \mathbb{A}(\mathbf{Z})$.

Theorem 7.4. *Suppose that* (C, ϕ) *is an* n*-dimensional visible symmetric Poincaré complex over* $\mathbb{A}(\mathbf{Z}[\pi])$ *and* (A, α, U) *a* $(\mathbf{Z}, m)$*-symmetric representation of* $\mathbf{Z}[\pi]$. *Then:*

$$\tau^{NEW}((C,\phi)\otimes(A,\alpha,U)) = U^!\tau^{NEW}(C,\phi)+\chi(C)\tau^{NEW}(\alpha) \in \hat{H}^{n+m}(\mathbf{Z}_2; K_1(\mathbf{Z})) = K_1(\mathbf{Z})$$

Proof. We make C into a signed complex by choosing some η_C. The morphism $\phi_0 \otimes (A, \alpha, U)$ is given by:

$$\phi_0 \otimes (A,\alpha,U) = S^m(\phi_0 \otimes (A,\alpha,U)) \circ \left(\bigoplus_{\mathrm{rank}_{\mathbf{Z}}(C^{n-r})} \alpha \right) : (S^m C \otimes (A,\alpha,U))^{n+m-r}$$

$$\to (S^m C \otimes (A,\alpha,U))_{r+m}$$

We regard the chain complex $S^m(C \otimes (A, \alpha, U))$ as the signed complex which one obtains after applying the functor $- \otimes (A, \alpha, U)$ to the signed complex C and suspending m times. We observe that $(S^m C \otimes (A, \alpha, U)^{n+2m-*} = S^m C^{n-*} \otimes (A, \alpha, U)$ as signed complexes. Hence:

$$\begin{aligned}
\tau^{NEW}((C,\phi)\otimes(A,\alpha,U)) &= \tau^{NEW}(S^m\phi_0 \otimes (A,\alpha,U)) + \tau^{NEW}(\bigoplus_r (-)^r \bigoplus_{\mathrm{rank}_{\mathbf{Z}}(C_r)} \alpha) \\
&= U^!\tau^{NEW}(\phi_0) + \chi(C)\tau^{NEW}(\alpha)
\end{aligned}$$

as required. □

Proof of Theorem 7.1. The theorem is known to be true if n is odd so we assume that $n = 2k$. From Theorem 7.4 we know that:

$$\tau^{NEW}((C,\phi)\otimes(A,\alpha,U)) = U^!\tau^{NEW}(C,\phi) + \chi(B)\tau^{NEW}(\alpha)$$

We know that the signature modulo four is determined by the absolute torsion 5.29 and the Euler characteristic. We also know that the signature is multiplicative for the untwisted product so it is sufficient to show that the right-hand side of the above equation does not depend on action U. In other words we want to show that

$$U^!\tau^{NEW}(C,\phi) = \mathrm{rank}(A).\epsilon.\tau(C,\phi) \tag{7.0.1}$$

The assumption that $\tau(C,\phi) = 0 \in \mathrm{Wh}(\mathbf{Z}[\pi])$ tells us that $\tau^{NEW}(C,\phi) = \tau(\pm g)$ for some $g \in \pi$. In this the two sides of the above equation differ by $\tau(U(g))$ so it will be sufficient to show that $\tau(U(g)) = 0$. We divide into two cases:

The case where n is even Suppose that $\tau(U(g)) = \tau(-1)$. We define a map $\lambda : \pi \to \mathbf{Z}_2$ by

$$g \mapsto \tau(U(g))$$

The subgroup $\ker(\lambda)$ is a normal subgroup of index 2. We can form the algebraic double-cover of (C,ϕ) associated to $\ker(\lambda)$; this is given by the twisted tensor product of (C,ϕ) with the form (B,β,V) with $B = \mathbf{Z}^2$, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$V(g) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } g \in \ker(\lambda) \\ \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } g \notin \ker(\lambda) \end{cases}$$

We write $V^! : K_1(\mathbf{Z}[\pi]) \to K_1(\mathbf{Z})$ for the transfer associated to this representation. The absolute torsion of this algebraic double-cover is given by:

$$\begin{aligned} \tau^{NEW}((C,\phi)\otimes(B,\beta,V)) &= V^!\tau^{NEW}(C,\phi) \\ &= V^!\tau(\pm g) \\ &= V^!\tau(\pm) + \tau(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \\ &= \tau(-1) + \tau(\pm) \end{aligned}$$

whereas the absolute torsion of the untwisted product is $\tau(\pm)$. Therefore the signature of the double cover associated to $\ker(\lambda)$ is not multiplicative modulo four. However, we know from [Wei92] Observation 8.2 that the signature of such a double cover of a visible symmetric complex is multiplicative modulo eight so we have a contradiction. Therefore $\tau(U(g)) = 0$ in this case.

The case where n is odd In this case the form α is anti-symmetric. Considering (A, α) to be a 2-dimensional symmetric Poincaré complex we see that $U(g)$ is a self-homotopy equivalence of such a complex. Therefore by corollary 5.28 $\tau(U(g)) = 0$.

$\square$

Chapter 8

The signature of a fibration modulo eight

In this chapter we will prove:

Theorem 8.1. *Let $F^{4m} \to E^{4n+4m} \to B^{4n}$ be a Poincaré fibration satisfying assumption 1.2 such that the action of $\pi_1(B)$ on $(H_{2m}(F;\mathbf{Z})/torsion) \otimes \mathbf{Z}_2$ is trivial. Then*

$$\mathrm{sign}(E) = \mathrm{sign}(F)\mathrm{sign}(B) \pmod 8$$

We will of course be proving this as a corollary of the analogous algebraic statement for twisted products. First some terminology: We say a $(\mathbf{Z}, m)$-symmetric representation (A, α, U) of a group ring $\mathbf{Z}[\pi]$ is *$\mathbf{Z}_2$-trivial* if

$$U(r) \otimes 1 = \epsilon(r) \otimes 1 : A \otimes \mathbf{Z}_2 \to A \otimes \mathbf{Z}_2$$

for all $r \in \mathbf{Z}[\pi]$. In the case of a fibration $F^{4m} \to E^{4n+4m} \to B^{4n}$ the $(\mathbf{Z}, 2m)$-symmetric representation (K, ϕ^F, U) is $\mathbf{Z}_2$-trivial if the action of $\pi_1(B)$ on

$$(H_{2m}(F;\mathbf{Z})/torsion) \otimes \mathbf{Z}_2$$

is trivial.

We can now state the algebraic analogue of the main Theorem:

Theorem 8.2. *Let (C, ϕ) be a $4n$-dimensional visible symmetric complex and let (A, α, U) be a $\mathbf{Z}_2$-trivial $(\mathbf{Z}, 2m)$-symmetric representation. Then*

$$\mathrm{sign}((A, \alpha, U) \otimes (C, \phi)) = \mathrm{sign}(C, \phi)\mathrm{sign}(A, \alpha) \pmod 8$$

The main idea behind the proof is to exploit the relationship between the signature modulo 8 and the Arf invariant of the Pontrjagin square established by Morita (Theorem 8.5). We will first show that the Theorem holds if the dimension of A is one by comparison with a double cover. We will then introduce a generalized Pontrjagin square for symmetric Poincaré complexes and we will relate this generalized Pontrjagin square on a symmetric complex to the usual Pontrjagin square on a twisted product. This will allow us to reduce to the case where (A, α, U) is the sum of one dimensional symmetric representations.

First we show how the main Theorem follows from the algebraic Theorem:

The proof of Theorem 8.1. Applying the symmetric construction to B yields a $4n$-dimensional visible symmetric complex $(C(\widetilde{B}), \phi^B)$. We have an $(\mathbf{Z}, 2m)$-symmetric representation (K, ϕ^F, U) of $\mathbf{Z}[\pi_1 B]$ given by the middle dimension of the fibre as in section 4.3. Moreover this representation is $\mathbf{Z}_2$-trivial. Theorem 4.9 states that the signature of E is the signature of $(C(\widetilde{B}), \phi^B) \otimes (K, \phi^F, U)$. The above Corollary tells us that modulo 8 this is the product of the signatures of $(C(\widetilde{B}), \phi^B)$ and (K, ϕ^F), which is the product of the signatures of F and B. □

8.1 The proof for one dimensional symmetric representations.

We first prove that Theorem 8.2 holds in the special case where $A = \mathbf{Z}$.

Proposition 8.3. *Let (C, ϕ) be a $4n$-dimensional visible symmetric complex and let $(\mathbf{Z}, \alpha, U)$ be a $\mathbf{Z}_2$-trivial $(\mathbf{Z}, 2m)$-symmetric representation. Then*

$$\operatorname{sign}((A, \alpha, U) \otimes (C, \phi)) = \operatorname{sign}(C, \phi)\operatorname{sign}(\mathbf{Z}, \alpha) \pmod 8$$

Proof. The argument is similar to that used in proving the modulo four result in the case where the dimension of the base space is a multiple of four, in that we can reduce to the case of a double cover and use the fact that the signature of such covers is known to be multiplicative modulo eight ([Wei92] Observation 7.6, [Ran92] Example 23.5C).

Clearly the symmetric form α is either 1 or -1; without loss of generality we assume that $\alpha = 1$. For each $g \in \pi$ the representation $U(g)$ is also either 1 or -1 - this induces a homomorphism $\lambda : \pi \to \mathbf{Z}_2$ given by

$$g \mapsto U(g) \in \{\pm 1\} = \mathbf{Z}_2$$

We can form the algebraic double-cover of (C, ϕ) associated to $\ker(\lambda)$; this is given by the twisted tensor product of (C, ϕ) with the $(\mathbf{Z}, 2m)$-symmetric representation

(B, β, V) with $B = \mathbf{Z}^2$, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$V(g) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } g \in \ker(\lambda) \\ \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } g \notin \ker(\lambda) \end{cases}$$

(c.f. the proof of Theorem 7.1 at the end of the previous chapter)

We now take the sum of $(\mathbf{Z}, 1, U)$ with a trivial $(\mathbf{Z}, 2m)$-symmetric representation $(\mathbf{Z}, 1, \epsilon)$ to form a $(\mathbf{Z}, 2m)$-symmetric representation $(\mathbf{Z}^2, 1 \oplus 1, U \oplus \epsilon)$. There is an obvious notion of an $(\mathbf{R}, 2m)$-symmetric representation given by replacing $\mathbf{Z}$ with $\mathbf{R}$ and we can form such a representation by tensoring a $(\mathbf{Z}, 2m)$-symmetric representation with $\mathbf{R}$ (we will omit the $\otimes \mathbf{R}$ from the maps U,V and β). After such tensoring there is an isomorphism between the $(\mathbf{R}, 2m)$-symmetric representations $(B \otimes \mathbf{R}, \beta, V)$ and $(\mathbf{R}^2, 1 \oplus 1, U \oplus \epsilon)$ given by:

$$\theta = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} : B \otimes \mathbf{R} \to \mathbf{R}^2$$

which satisfies

$$(1 \oplus 1) = \theta \circ \beta \circ \theta^* : \mathbf{R}^2 \to \mathbf{R}^2$$

and

$$\theta \circ V(r) = (U(r) \oplus \epsilon(r)) \circ \theta : B \otimes \mathbf{R} \to \mathbf{R}^2$$

Hence for each r we have a commutative diagram:

$$\begin{tikzcd}
(C_{2n-r} \otimes (B \otimes \mathbf{R}, \beta, V))^* \arrow[d, "\bigoplus \beta"] & (C_{2n-r} \otimes (\mathbf{R}^2, 1 \oplus 1, U \oplus \epsilon))^* \arrow[l, "\bigoplus \theta^*"'] \arrow[d, "\bigoplus (1 \oplus 1)"] \\
C^{2n-r} \otimes (B \otimes \mathbf{R}, \beta, V) \arrow[r, "\bigoplus \theta"] \arrow[d, "\phi_0 \otimes (B \otimes \mathbf{R}, \beta, V)"] & C^{2n-r} \otimes (\mathbf{R}^2, 1 \oplus 1, U \oplus \epsilon) \arrow[d, "\phi_0 \otimes (\mathbf{R}^2, 1 \oplus 1, U \oplus \epsilon)"] \\
C_r \otimes (B \otimes \mathbf{R}, \beta, V) \arrow[r, "\bigoplus \theta"] & C_r \otimes (\mathbf{R}^2, 1 \oplus 1, U \oplus \epsilon)
\end{tikzcd}$$

Consequently θ induces an isomorphism of symmetric complexes:

$$\theta_* : (C, \phi) \otimes (B \otimes \mathbf{R}, \beta, V) \to (C, \phi) \otimes (\mathbf{R}^2, 1 \oplus 1, U \oplus \epsilon)$$

Therefore these complexes have the same signature and hence

$$\operatorname{sign}((C, \phi) \otimes (B, \beta, V)) = \operatorname{sign}((C, \phi) \otimes (\mathbf{Z}^2, 1 \oplus 1, U \oplus \epsilon))$$

However we know that the signature of the double cover of a visible symmetric complex is multiplicative modulo eight hence the left-hand side is equal to

2sign(C, ϕ) modulo eight (see [Wei92] Observation 7.6, [Ran92] Example 23.5C). The signature of the right-hand side is equal to sum of the signatures of (C, ϕ) and $(C, \phi) \otimes (\mathbf{Z}, 1, U)$. Hence

$$\mathrm{sign}((C, \phi) \otimes (\mathbf{Z}, 1, U)) = \mathrm{sign}(C, \phi) = \mathrm{sign}(C, \phi)\mathrm{sign}(\mathbf{Z}, \alpha) \pmod 8$$

as required. □

8.2 The Pontrjagin square and a theorem of Morita.

The Pontrjagin square (see [BT62]) is an unstable cohomology operation:

$$\mathcal{P} : H^k(X; \mathbf{Z}_{2a}) \to H^{2k}(X; \mathbf{Z}_{4a})$$

For a $2k$-dimensional symmetric complex (C, ϕ), there is defined a Pontrjagin square:

$$\mathcal{P}_{(C,\phi)} : H^k(C; \mathbf{Z}_{2a}) \to \mathbf{Z}_{4a} \ ; \ x^* \mapsto \langle x^*, (\phi_0 + d\phi_1)(x^*) \rangle$$

where x^* is a cochain representing $[x^*] \in H^k(C; \mathbf{Z}_{2a})$. This is related to the traditional topological Pontrjagin square in the following way: For a space X and homology class $[X] \in H_{2k}(X)$, the symmetric construction of Ranicki [Ran80b] yields a $2k$-dimensional symmetric complex $(C(X), \Delta[X])$. Then for all $x \in H^k(X; \mathbf{Z}_{2a}) = H^k(C(X); \mathbf{Z}_{2a})$ we have

$$\mathcal{P}_{(C(X),\Delta[X])} = \langle \mathcal{P}(x), [X] \rangle \in \mathbf{Z}_{4a}$$

The Pontrjagin square is an example of a quadratic enhancement of a bilinear form on a $\mathbf{Z}_2$-vector space ([Mor71], [Tay]).

Definition 8.4. *Let $\mu : V \otimes V \to \mathbf{Z}_2$ be a non-singular symmetric form on a $\mathbf{Z}_2$-vector space V. A function $\eta : V \to \mathbf{Z}_4$ is said to be a* quadratic enhancement *of μ if*

$$\eta(x + y) = \eta(x) + \eta(y) + 2\mu(x \otimes y) \in \mathbf{Z}_4$$

where $2 : \mathbf{Z}_2 \to \mathbf{Z}_4$ is the inclusion.

If (C, ϕ) is Poincaré then the Pontrjagin square is a quadratic enhancement of the form on $H^k(C; \mathbf{Z}_2)$ given on the cochain level by $(x \otimes y) \mapsto \langle x, \phi_0(y) \rangle$.

We define the Arf invariant Arf(η) of a quadratic enhancement as follows ([Mor71], [Tay], [Bro72]): The *Gauss sum* $G(\eta) \in \mathbf{C}$ is defined to be:

$$G(\eta) = \sum_{x \in V} i^{\eta(x)}$$

It can be shown that $G(\eta)$ is non-zero and that furthermore the argument of $G(\eta)$ is a multiple of $\frac{\pi}{4}$. We define the *Arf invariant* $\mathrm{Arf}(\eta) \in \mathbf{Z}_8$ to be such that the argument of $G(\eta)$ is $\mathrm{Arf}(\eta)\frac{\pi}{4}$.

We have the following theorem of Morita [Mor71], [Tay]:

Theorem 8.5. *Let (C,ϕ) be a $4k$-dimensional symmetric Poincaré complex over $\mathbf{Z}$. Then the Arf invariant of $\mathcal{P}_{(C,\phi),2\mathbf{Z}} : H^k(C;\mathbf{Z}_2) \to \mathbf{Z}_4$ satisfies*

$$\mathrm{Arf}(\mathcal{P}_{(C,\phi),2\mathbf{Z}}) = \mathrm{sign}(C,\phi) \pmod 8$$

Proof. The original proof of Morita [Mor71] holds in the more general case of symmetric Poincaré complexes, as does the proof of Taylor [Tay]. See also the recent work of Ranicki and Taylor [RT05]. □

8.3 The generalized Pontrjagin square.

We now generalize this to the setting of a symmetric complex over a ring with involution R.

Definition 8.6. 1. *A* *-invariant ideal *I in a ring with involution R is a left ideal I satisfying $I^* = I$. Note that such an ideal is also a right ideal. The involution on R extends to an involution on R/I in the obvious way.*

2. *Given a $2k$-dimensional symmetric complex (C,ϕ) over R and a *-invariant ideal I then we define the Pontrjagin square to be the map*

$$\mathcal{P}_{(C,\phi),I} : H^k(C;R/I) \to R/(I^2+2I)\ ;\ x \mapsto \langle x, (\phi_0 + d\phi_1)(x)\rangle$$

where x is a cochain representative of $[x] \in H^k(C;R/I)$. One can easily check that this map is well-defined.

By $H^k(C;R/I)$ we mean $H^k(C \otimes_R R/I)$. If $R = \mathbf{Z}$ and $I = 2a\mathbf{Z}$ then this coincides with the Pontrjagin square defined above. We list a few important properties of the generalized Pontrjagin square analogous to those of the usual Pontrjagin square:

Proposition 8.7. 1. *$\mathcal{P}_{(C,\phi),I}(rx) = r\mathcal{P}_{(C,\phi),I}(x)r^*$*

2. *$\mathcal{P}_{(C,\phi),I}(x+y) = \mathcal{P}_{(C,\phi),I}(x) + \mathcal{P}_{(C,\phi),I}(y) + \langle x, \phi_0(y)\rangle + (-)^k\langle x, \phi_0(y)\rangle^*$*

3. *$\mathcal{P}_{(C,\phi),I}(x) = \langle x, \phi_0(x)\rangle \in R/I$*

4. $\mathcal{P}_{(C,\phi),I}(x)^* = (-)^k \mathcal{P}_{(C,\phi),I}(x)$

Proof. Easy manipulations. □

We will have a particular situation in mind. Suppose that $R = \mathbf{Z}[\pi]$ and $I = \epsilon^{-1}(2\mathbf{Z})$. Then the generalized Pontrjagin square is a map:

$$\mathcal{P}_{(C,\phi),I} : H^k(C;\mathbf{Z}_2) \to \mathbf{Z}[\pi]/(I^2)$$

since the ideal I contains two. We can describe the right-hand side more explicitly: The induced augmentation map $\bar{\epsilon} : \mathbf{Z}[\pi]/(I^2) \to \mathbf{Z}_4$ splits so we can decompose $\mathbf{Z}[\pi]/(I^2)$ as:

$$\mathbf{Z}[\pi]/(I^2) = \mathbf{Z}_4 \oplus \bar{\epsilon}^{-1}(0)$$

We have a basis for $\epsilon^{-1}(0)$ given by the elements $1-g$ for $g \in \pi$. The relations in $\bar{\epsilon}^{-1}(0)$ are given by the products

$$0 = (1-g)(1-h) = 1-g-h+gh$$

or in other words:

$$(1-g)+(1-h) = (1-gh).$$

Therefore we have a map

$$\begin{aligned} \theta : \ \pi &\to \bar{\epsilon}^{-1}(0) \\ g &\mapsto (1-g) \end{aligned}$$

The group $\bar{\epsilon}^{-1}(0)$ is abelian and consists only of 2-torsion so the map θ factors through $\pi_{ab} \otimes \mathbf{Z}_2$. Examining the relations in $\bar{\epsilon}^{-1}(0)$ tells us that the induced map

$$\bar{\theta} : \pi_{ab} \otimes \mathbf{Z}_2 \to \bar{\epsilon}^{-1}(0)$$

is an isomorphism. The Hurewicz Theorem and the Universal Coefficient Theorem combine to tells us that $\pi_{ab} \otimes \mathbf{Z}_2$ is isomorphic to $H_1(\pi;\mathbf{Z}_2)$. Putting all this together we see that:

$$\mathbf{Z}[\pi]/(I^2) \cong \mathbf{Z}_4 \oplus H_1(\pi;\mathbf{Z}_2) \tag{8.3.1}$$

in a natural way.

8.4 The generalized Pontrjagin square on visible symmetric complexes.

The results of this section have been withdrawn by the author. A new argument is used in the next section to prove the modulo eight result.

8.5 The generalized Pontrjagin square and twisted products.

We now fix our ring to be a group ring $\mathbf{Z}[\pi]$ and I the ideal given by $I = \epsilon^{-1}(2\mathbf{Z})$. If (A, α, U) is $\mathbf{Z}_2$-trivial we have a natural identification:

$$H^{2n+2m}(C \otimes (A, \alpha, U); \mathbf{Z}_2) \cong H^{2n}(C; \mathbf{Z}_2) \otimes_{\mathbf{Z}_2} A \cong H^{2n+2m}(C \otimes A; \mathbf{Z}_2)$$

The following technical lemma describes how the generalized Pontrjagin square behaves on twisted products:

Lemma 8.8. *Let (C, ϕ) be a $4n$-dimensional symmetric complex and (A, α, U) be a $\mathbf{Z}_2$-trivial $(\mathbf{Z}, 2m)$-symmetric representation. Then the Pontrjagin square evaluated on $c \otimes a$, for $a \in A$ and c a $2n$-cochain in C is given by:*

$$\mathcal{P}_{(C,\phi)\otimes(A,\alpha,U),2\mathbf{Z}}(c \otimes a) = \langle a, U(\mathcal{P}_{(C,\phi),I}(c)) \circ \alpha(a)\rangle \in \mathbf{Z}_4$$

Note that the map U is applied to the Pontrjagin square in the formula on the right hand side; by this we mean U applied to some representative of $\mathcal{P}_{(C,\phi),I}(c)$ in $\mathbf{Z}[\pi]$. One can easily check that the RHS is well-defined. Note also that this proposition applies only to elements of the form $c \otimes a \in C^{2n} \otimes (A, \alpha, U)$.

Proof.

$$\begin{aligned}
\mathcal{P}_{(C,\phi)\otimes(A,\alpha,U)} &= \langle c \otimes a, ((\phi_0 + \phi_1 d^*) \otimes \alpha)(c \otimes a)\rangle \\
&= \langle a, \alpha \circ U(\langle c, (\phi_0 + \phi_1 d^*)c\rangle^*)(a)\rangle \\
&= \langle a, U(\langle c, (\phi_0 + \phi_1 d^*)c\rangle) \circ \alpha(a)\rangle \\
&= \langle a, U(\mathcal{P}_{(C,\phi),I}(c)) \circ \alpha(a)\rangle \in \mathbf{Z}/4i\mathbf{Z}
\end{aligned}$$

□

The proof of Theorem 8.2. Without loss of generality we may assume that we have some choice of basis $\{a_i\}$ for A such that the form α is diagonal with respect to this basis. We can represent any element of $H^{2n+2m}(C \otimes (A, \alpha, U))$ as a sum $\sum_i c_i \otimes a_i$ for some $c_i \in C^{2n+2m}$. Using the additivity property of the Pontrjagin square, the above lemma and the fact that (A, α, U) is $\mathbf{Z}_2$-trivial we see that:

$$\begin{aligned}
\mathcal{P}_{(C,\phi)\otimes(A,\alpha,U),2\mathbf{Z}}(\sum_i c_i \otimes a_i) &= \sum_i \langle a_i, U(\mathcal{P}_{(C,\phi),I}(c_i)) \circ \alpha(a_i)\rangle \\
&\quad + \sum_{j>k} 2\langle a_j, \alpha(a_k)\rangle \epsilon(\langle c_j, \phi_0(c_k)\rangle)) \\
&= \sum_i \langle a_i, U(\mathcal{P}_{(C,\phi),I}(c_i)) \circ \alpha(a_i)\rangle
\end{aligned}$$

From this expression we see that the Pontrjagin square on the twisted product only depends on the diagonal elements of each $U(r)$ with respect to the basis $\{a_i\}$. For each i we have a homomorphism $w_i : \pi \to \{\pm 1\} = \mathbf{Z}_2$ given by

$$w_i(g) = U(g)_{i,i} \pmod 4$$

with $U(g)_{i,i}$ being the diagonal elements of U with respect to the basis. These maps extend linearly to maps $W_i : \mathbf{Z}[\pi] \to \mathbf{Z}$ in the obvious way and we may construct a representation $W \to \mathrm{Hom}(A, A)^{op}$ given by:

$$W(r) = \begin{pmatrix} W_1(r) & & & \\ & W_2(r) & & \\ & & \ddots & \\ & & & W_n(R) \end{pmatrix} : A = \mathbf{Z}^n \to A$$

Clearly (A, α, W) is a $\mathbf{Z}_2$-trivial $(\mathbf{Z}, 2m)$-symmetric representation. Moreover since the diagonal terms of U coincide with those of W modulo four we see that the Pontrjagin square on the twisted products $(C, \phi) \otimes (A, \alpha, U)$ and $(C, \phi) \otimes (A, \alpha, W)$ coincide. The Theorem of Morita 8.5 tells us that

$$\mathrm{sign}((C, \phi) \otimes (A, \alpha, U)) = \mathrm{sign}((C, \phi) \otimes (A, \alpha, W)) \pmod 8$$

But the symmetric representation (A, α, W) splits into a sum $(A_i, \alpha_{i,i}, W_i)$ with each $A_i = \mathbf{Z}$. Hence

$$\mathrm{sign}((C, \phi) \otimes (A, \alpha, U)) = \sum_i \mathrm{sign}((C, \phi) \otimes (A_i, \alpha_{i,i}, W_i)) \pmod 8$$

However by proposition 8.3 the signature is multiplicative modulo eight for the twisted products on the right-hand side. Therefore

$$\begin{aligned} \mathrm{sign}((C, \phi) \otimes (A, \alpha, U)) &= \sum_i \mathrm{sign}((C, \phi) \otimes (A_i, \alpha_{i,i}, W_i)) \\ &= \sum_i \mathrm{sign}(C, \phi)\mathrm{sign}(A_i, \alpha_{i,i}) \\ &= \mathrm{sign}(C, \phi)\mathrm{sign}(A, \alpha) \pmod 8 \end{aligned}$$

as required. □

The following example illustrates that the "visible" condition is required (see also [Ran81] Proposition 7.6.8): Let $\pi = \mathbf{Z}_2$ generated by t and define the 0-dimensional symmetric Poincaré complex (C, ϕ) by $C_0 = \mathbf{Z}[\mathbf{Z}_2]$ and

$$\phi_0 = t : C^0 = \mathbf{Z}[\mathbf{Z}_2] \to C_0 = \mathbf{Z}[\mathbf{Z}_2]$$

We know that this complex is not visible because of lemma 2.7. Let (A, α, U) be the $(\mathbf{Z}, 0)$-symmetric representation defined by $A = \mathbf{Z}$, $\alpha = 1 : A^* \to A$ and $U(t) = -1 : A \to A$. Then the signature of the untwisted product is 1 whereas the signature of the twisted product is -1. Note that this example shows that the modulo four result is false without the "visible" assumption.

Bibliography

[Ati69] M. F. Atiyah. The signature of fibre-bundles. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 73–84. Univ. Tokyo Press, Tokyo, 1969.

[Bro68] R. Brown. *Elements of modern topology.* McGraw-Hill Book Co., New York, 1968.

[Bro72] E. H. Brown, Jr. Generalizations of the Kervaire invariant. *Ann. of Math. (2)*, 95:368–383, 1972.

[BT62] W. Browder and E. Thomas. Axioms for the generalized Pontryagin cohomology operatons. *Quart. J. Math. Oxford. Ser. (2)*, 13:55–60, 1962.

[CHS57] S. S. Chern, F. Hirzebruch, and J.-P. Serre. On the index of a fibered manifold. *Proc. Amer. Math. Soc.*, 8:587–596, 1957.

[End98] Hisaaki Endo. A construction of surface bundles over surfaces with non-zero signature. *Osaka J. Math.*, 35(4):915–930, 1998.

[Got79] Daniel Henry Gottlieb. Poincaré duality and fibrations. *Proc. Amer. Math. Soc.*, 76(1):148–150, 1979.

[Hir69] F. Hirzebruch. The signature of ramified coverings. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 253–265. Univ. Tokyo Press, Tokyo, 1969.

[HKR05] I. Hambleton, A. J. Korzeniewski, and A. Ŕanicki. The signature of a fibre bundle is multiplicative mod 4. *Preprint*, 2005.

[HNK71] F. Hirzebruch, W. D. Neumann, and S. S. Koh. *Differentiable manifolds and quadratic forms.* Marcel Dekker Inc., New York, 1971. Appendix II by W. Scharlau, Lecture Notes in Pure and Applied Mathematics, Vol. 4.

[HRT87] I. Hambleton, A. Ranicki, and L. Taylor. Round L-theory. *J. Pure Appl. Algebra*, 47(2):131–154, 1987.

[Ker56] M. Kervaire. Courbure intégrale généralisée et homotopie. *Math. Ann.*, 131:219–252, 1956.

[KKNO73] U. Karras, M. Kreck, W. D. Neumann, and E. Ossa. *Cutting and pasting of manifolds; SK-groups*. Publish or Perish Inc., Boston, Mass., 1973. Mathematics Lecture Series, No. 1.

[Kod67] K. Kodaira. A certain type of irregular algebraic surfaces. *J. Analyse Math.*, 19:207–215, 1967.

[Kor05] A. J. Korzeniewski. Absolute whitehead torsion. *Preprint*, 2005.

[KT03] S. Klaus and P. Teichner. Private communications. 2003.

[LMP69] G. Lusztig, J. Milnor, and F. P. Peterson. Semi-characteristics and cobordism. *Topology*, 8:357–359, 1969.

[LR88] W. Lück and A. Ranicki. Surgery transfer. In *Algebraic topology and transformation groups (Göttingen, 1987)*, volume 1361 of *Lecture Notes in Math.*, pages 167–246. Springer, Berlin, 1988.

[LR92] W. Lück and A. Ranicki. Surgery obstructions of fibre bundles. *J. Pure Appl. Algebra*, 81(2):139–189, 1992.

[Lüc86] W. Lück. The transfer maps induced in the algebraic K_0-and K_1-groups by a fibration. I. *Math. Scand.*, 59(1):93–121, 1986.

[Mey72] W. Meyer. Die Signatur von lokalen Koeffizientensystemen und Faserbündeln. *Bonn. Math. Schr.*, (53):viii+59, 1972.

[Mey73] Werner Meyer. Die Signatur von Flächenbündeln. *Math. Ann.*, 201:239–264, 1973.

[Mil66] J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.*, 72:358–426, 1966.

[Mor71] S. Morita. On the Pontrjagin square and the signature. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 18:405–414, 1971.

[MT68] Robert E. Mosher and Martin C. Tangora. *Cohomology operations and applications in homotopy theory*. Harper & Row Publishers, New York, 1968.

[Neu78] Walter D. Neumann. Multiplicativity of signature. *J. Pure Appl. Algebra*, 13(1):19–31, 1978.

[Ped82] Erik Kjaer Pedersen. Geometrically defined transfers, comparisons. *Math. Z.*, 180(4):535–544, 1982.

[Qui70] Frank Quinn. A geometric formulation of surgery. In *Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969)*, pages 500–511. Markham, Chicago, Ill., 1970.

[Ran80a] A. Ranicki. The algebraic theory of surgery. I. Foundations. *Proc. London Math. Soc. (3)*, 40(1):87–192, 1980.

[Ran80b] A. Ranicki. The algebraic theory of surgery. II. Applications to topology. *Proc. London Math. Soc. (3)*, 40(2):193–283, 1980.

[Ran81] Andrew Ranicki. *Exact sequences in the algebraic theory of surgery*, volume 26 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1981.

[Ran85] A. Ranicki. The algebraic theory of torsion I. Foundations. In *Proc. 1983 Rutgers Topology Conf. on Algebraic and Geometric Topology*, volume 1126 of *Lecture Notes in Math.*, pages 199–237. Springer, Berlin, 1985.

[Ran89] A. Ranicki. Additive L-theory. *K-Theory*, 3(2):163–195, 1989.

[Ran92] A. A. Ranicki. *Algebraic L-theory and topological manifolds*, volume 102 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1992. Available from `http://www.maths.ed.ac.uk/~aar/books/topman.pdf`.

[Ran98] A. Ranicki. *High-dimensional knot theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 1998. Algebraic surgery in codimension 2, With an appendix by Elmar Winkelnkemper.

[RT05] A. Ranicki and L. Taylor. The mod 8 signatures of normal spaces. *In preparation.*, 2005.

[Tay] L. Taylor. Gauss sums in algebra and topology. *Preprint, available at* `http://www.nd.edu/~taylor/papers/Gauss_sums.pdf`.

[Wal99] C. T. C. Wall. *Surgery on compact manifolds*, volume 69 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 1999. Edited and with a foreword by A. A. Ranicki.

[Wei92] M. Weiss. Visible L-theory. *Forum Math.*, 4(5):465–498, 1992.

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